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SPHERICAL HARMONIC EXPANSION OF THE LEVITUS SEA SURFACE TOPOGRAPHY

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by

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Abstract

Prior information for the stationary sea surface topography (SST) may be needed in altimetric solutions that intend to simultaneously improve the gravity field and determine the SST. For this purpose the oceanographically derived SST estimates are represented by a spherical harmonic expansion. The spherical harmonic coefficients are computed from a least squares adjustment of the data covering the majority of the oceanic regions of the world. Several tests are made to determine the optimum maximum degree of solution and the best configuration of the geometry of the data in order to obtain a solution that fits the data and also provides a good spectral representation of the SST.

Foreword

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Table of Contents

Abstract ii
Foreword iii
Introduction 1
The Levitus SST 2
Harmonic Analysis of a Function on a Sphere 4
Determination of the SST Spherical Harmonic Coefficients 7
Conclusions 10
References 11
Appendix A - Sea Surface Topography Maps 12
Appendix B - Spherical Harmonic Coefficients from Different Solutions of
Levitus SST Estimates 27

Introduction.

One of the main problems in marine geodesy is the determination of the geoid from the sea surface heights that are computed from satellite altimetric observations. In order to do that independent estimates of the stationary and time variable sea surface topography (SST) need to be provided. Time variations of SST can be computed in a rather straightforward way by analysing the overlapping tracks during the repeat era of a satellite altimetric mission (Cheney et al., 1983). These variations can then be removed from the sea surface heights. Alternatively they can be considered as noise during the processing of altimeter data with traditional crossover techniques and therefore be filtered out from the sea surface heights (e.g. Rowlands, 1981). Estimates of the stationary SST, on the other hand, that has an expected total variation of about 2 meters, can be provided by oceanographic methods (e.g. Levitus, 1982) using observations of ocean temperature, salinity and oxygen content and imposing geostrophic conditions in solving the equations of motion for the oceans.

Stationary SST can also be computed by traditional geodetic techniques. In one such determination (Engelis, 1985) geoid undulations computed from a low degree satellite derived gravity field, are subtracted from the sea surface heights computed from altimetry. Because of the errors in the determination of the satellite derived gravity field only the long wavelength part of these reduced sea surface heights can represent the stationary SST with some degree of confidence. In order to determine this long wavelength SST an harmonic analysis of the reduced heights is required. Then a low pass filtering is performed to retain only the low degree coefficients that have a favourable signal to noise ratio.

Recently there have been attempts to simultaneously determine the long wavelength SST and improve the gravity field of the earth. One such method incorporates a low degree (n_{max} = 10) spherical harmonic model into a general dynamic solution for a low degree (n_{max} = 50) gravity field of the earth, currently being attempted at NASA Goddard Space Flight Center. In such a solution observations to geodetic satellites, altimeter data and terrestrial gravity anomalies are used. In an alternative method being proposed by Engelis (1987), altimeter observations are used in a combined solution to reduce the radial orbit error, improve the geoid and determine the stationary SST.

The effectiveness of all geodetic techniques to determine the SST is subject to the accurate representation of the spectral content of the SST itself, or, in other words, subject to the correct implementation of the spherical harmonic models that are used. Particularly for the last method, where apriori SST information by wavelength (i.e. degree variances) is needed, it is important that some estimates of the spectral behavior of SST is available. Such estimates have been provided in the past by Engelis (1985) who has used the Levitus data to estimate harmonic coefficients and their degree variances. In that estimation the orthogonality principle assuming data all over the sphere was used. In the present investigation this determination is reexamined since the SST is not a complete function on the sphere but is only defined in oceanic areas. Therefore, consideration of the SST as a global function introduces problems, the most important of which is that, the resulting harmonic coefficients have a lower power since they are forced to also fit the land regions which are filled with SST values that are traditionally considered to be zeroes. In the present analysis the Levitus data, only in the ocean areas, will be used to determine spherical harmonic coefficients. These coefficients can provide, hopefully, the best reference values needed for the combined solutions. A more general purpose of this analysis is to examine the problems involved with the spectral decomposition of data that are not globally distributed on the sphere.

The Levitus SST.

In oceanographic methods to determine the stationary SST, temperature, salinity and disolved oxygen content of the oceans are used to determine pressure and water density, which in turn are used to solve the geostrophic equations of motion in the oceans. In order to do that a reference equipotential surface (surface of no motion), which ideally can be the geoid, must be used. Due to the inability to use an estimate of the geoid, a deep surface is defined to be a surface of no motion. Then solution of the equations of motion provides the mean annual anomaly of the geopotential thickness of the layer between that surface and the ocean surface.

The most recent determination of such a dynamic SST is made by Levitus (1982) who used data from the National Oceanographic Data Center. A first analysis of the data, made by Levitus, indicated several problems. The most important problem was regional biases in the data and lack of data in extended regions. Moreover there were temporal representation problems in the data since observations were not synoptic but scattered with respect to time (with the exception of a few limited areas) and so, the results cannot in a strict sense be considered a true long term average.

After the initial stability and statistical checks to eliminate spurious observations, averages of data in 1°x1° blocks were created. In order to overcome biases and lack of data a smoothing operation was performed. This smoothing consisted of a weighted average operation in which a Gaussian type filter was used as a weighting operator. The radii of the Gaussian filter ranged from 1540 km to 770 km depending on the region. As a result, any signal with wavelengths less than 800 km was eliminated while wavelengths between 800 km and 3000 km were affected with changes in the amplitudes of the signal. For example wavelengths of 1000 km had a reduction in amplitude of at least 50% (Levitus, 1982, Figure 11). The minimum wavelengths of 800 km roughly correspond to a maximum degree 20 in a spherical harmonic expansion. Levitus considers that the resulting large scale features are representative of the real ocean, although it is expected that some local differences can occur because of interannual variability. This smoothed data set was used to compute the annual mean anomaly of geopotential thickness of several layers corresponding to different deep surfaces considered to be surfaces of no motion.

The dynamic topography that is used in the present analysis is the one with respect to a 2250 db surface. This data set consists of 33856 1°x1° mean values in the ocean areas of the world. Their spatial distribution is shown in Figure 1 of Appendix A. A first analysis of this data set has indicated some outliers in the west equatorial region of the Pacific ocean. After these values are rejected, the weighted mean value of the SST set is computed to be 2.02

meters. This mean is removed, since for geodetic applications any such terms are absorbed by the mean earth ellipsoid that is used to reference the geoid. Analysis of the centered quantities indicates that the SST estimates range from -1.40 meters at the southernmost latitudes close to Antartica, to about 1 meter in the northern Pacific. A notable exception to the above range has been found in the Mediterrannean sea where there is a sparse coverage of 82 $1^\circ x 1^\circ$ values all of them having magnitudes smaller than -3 meters. Additionally in the oceanic regions above 70° as well as in the region between Greenland and Scandinavia there are 3134 SST estimates with a mean value of -1.80 meters and a very small, almost latitudinal variation. These values differ from neighboring values in the North Atlantic region as well as values in the Northern Pacific by as much as 1 meter.

This original data set with the mean value removed is the one that was used in Engelis (1985). In expanding the SST into spherical harmonics one is interested in determining a set of coefficients that best represent the oceanic regions within latitude limits that are also attained by satellite altimetry. Therefore, it was decided (Rapp, 1985, Appendix B) to reject all the 3134 values in the northernmost latitudes. Furthermore, the Levitus estimates in the Mediterrannean sea were replaced by estimates from a map by Lisitzin (Lisitzin 1974, p.153) that was given with respect to a 4000 db surface. In order to put the Lisitzin estimates into the same reference system as the Levitus set, their mean value was removed and the global mean value of the Levitus set was added.

In the present investigation four sets of SST estimates are used in order to see what is the effect of particular regions on the global analysis of SST. These sets are the following.

1. Original Levitus data set centered around its mean value of 2.02 meters and containing 33856 estimates with an RMS value of 80.3 cm (SET1).

2. Same as SET1 but augmented by the Lisitzin estimates in the Mediterrannean sea, centered around its mean value of 2.02 meters and containing 34056 estimates with an RMS value of 78.8 cm (SET2).

3. Same as SET2 but without the 3134 estimates in the northernmost latitudes. There is a total of 30922 estimates with their mean value of 2.01 meters removed and an RMS value of 62.4 cm (SET3).

4. Same as SET3 but with no data in the Mediterrannean sea. There are 30640 estimates with their mean value of 2.03 meters removed and an RMS value of 62.4 cm (SET4).

The spatial distribution of all four data sets is shown in Figures 1-4 of Appendix A.

Harmonic Analysis of a Function on a Sphere.

A square integrable analytical function $f(\phi,\lambda)$ defined on a unit sphere $-\pi/2<\phi<\pi/2$ and $0\leq\lambda\leq 2\pi$ can be expanded in a series of surface spherical harmonics

$$f(\phi,\lambda) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{a=0}^{1} \bar{C}^{a}_{\ell m} \bar{Y}^{a}_{\ell m} (\phi,\lambda) \tag{1}$$

where

$$\bar{C}^{a}_{\ell m} = \begin{cases} \bar{C}_{\ell m} & \text{if } a=0 \\ \bar{S}_{\ell m} & \text{if } a=1 \end{cases} \tag{2}$$

and

$$\bar{Y}^{a}_{\ell m} = \begin{cases} \bar{P}_{\ell m}(\sin\phi)\cos m\lambda & \text{if } a=0 \\ \bar{P}_{\ell m}(\sin\phi)\sin m\lambda & \text{if } a=1 \end{cases} \tag{3}$$

$\bar{C}_{\ell m}$, $\bar{S}_{\ell m}$ are the fully normalized spherical harmonic coefficients of the function $f(\phi,\lambda)$ and $\bar{P}_{\ell m}(\sin\phi)$ are the fully normalized associated Legendre functions of the first kind, such that

$$\frac{1}{4\pi} \int \bar{Y}^{a}_{\ell m}\bar{Y}^{b}_{ij}\,d\sigma = \begin{cases} 1 & \text{if } \ell=i,\ m=j,\ a=b \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Using the orthogonality principle that is expressed by equation (4) the spherical harmonic coefficients can be derived from

$$\bar{C}^{a}_{\ell m} = \frac{1}{4\pi} \int f(\phi,\lambda)\ \bar{Y}^{a}_{\ell m}(\phi,\lambda)\,d\sigma \tag{5}$$

These coefficients are independent and they form a basis of the function on the sphere. In terms of linear algebra, equation (1) is the spectral decomposition of the function $f(\phi,\lambda)$, $\bar{Y}_{\ell m}$ are the eigenfunctions and $\bar{C}_{\ell m}$ are the eigenvalues of the function.

When discrete point realizations of the function are given on the sphere in an equiangular gridded form, with grid intervals $\Delta\phi$, $\Delta\lambda$ along latitude and longitude respectively, then the upper limit of equation (1) changes from infinity to some maximum degree ℓ_{max} that corresponds to the Nyquist frequency of the sampled set and is equal to $\pi/\Delta\lambda$. Furthermore, equation (5) becomes

$$\bar{C}^{a}_{\ell m} = \frac{1}{4\pi} \sum_{i=0}^{L-1} \sum_{j=0}^{2L-1} f(\phi_i,\lambda_j)\bar{Y}^{a}_{\ell m}(\phi_i,\lambda_j)\sigma_{ij} \tag{6}$$

where σ_{ij} is the finite surface element and is equal to

$$\sigma_{ij} = \Delta\lambda(\sin(\phi_i+\Delta\phi)-\sin\phi_i) \tag{7}$$

In equation (6), L is the number of gridded samples along a parallel and is equal to ℓ_{max}. The orthogonality principle is still valid and independent coefficients can be obtained up to ℓ_{max}. These coefficients are insensitive to the maximum degree of solution as long as this is smaller than ℓ_{max} since folding of frequencies is prevented by the orthogonality of the harmonics. Attempts to determine coefficients of higher degrees result into aliased estimates that are fully correlated with coefficients of degrees lower than ℓ_{max}.

When the function $f(\phi,\lambda)$ is sampled in the form of equiangular area means, things become a little more complicated. Equation (1) becomes

$$f(\phi_i,\lambda_j) = \frac{1}{\sigma_{ij}} \sum_{\ell=0}^{\ell_{max}} \sum_{m=0}^{\ell} \sum_{a=0}^{1} \bar{C}^a_{\ell m} \int_{\sigma_{ij}} \bar{Y}^a_{\ell m} d\sigma \tag{8}$$

or, in an expanded form

$$f(\phi_i,\lambda_j) = \frac{1}{\sigma_{ij}} \sum_{\ell=0}^{\ell_{max}} \sum_{m=0}^{\ell} (\bar{C}_{\ell m} IC_m + \bar{S}_{\ell m} IS_m) IP_{\ell m} \tag{9}$$

where IC_m, IS_m, $IP_{\ell m}$ are the integrated cosines and sines of order m and the integrated associated Legendre functions, respectively. The maximum degree of expansion is again specified by the relationship $\ell_{max} = \pi/\Delta\lambda$ where $\Delta\lambda$ is now the size of the block.

The problem in expanding area means into spherical harmonics is that the orthogonality principle is not valid anymore. As a matter of fact deviations from the orthogonality principle are a function of the block size. These deviations tend to zero as the block size tends to zero. For a given block size deviations from the orthogonality principle increase with increasing degree. Approximations to the orthogonality principle and therefore estimates of the spherical harmonic coefficients can be obtained by introducing the so called desmoothing factors (Colombo, 1981, Rapp, 1986). Then equation (5) becomes

$$\bar{C}^a_{\ell m} = \frac{1}{4\pi q_\ell} \sum_{i=0}^{L-1} \sum_{j=0}^{2L-1} f(\phi_i \lambda_j) \int_{\sigma_{ij}} \bar{Y}^a_{\ell m}(\phi_i,\lambda_j) d\sigma \tag{10}$$

where q_ℓ can be either the Pellinen operator or an optimum quadrature weight defined by Colombo (1981). With the incorporation of the desmoothing factor errors due to the finite size of the block and errors arising from the dampening of higher frequencies because of the averaging to generate the mean values, are reduced.

Spherical harmonic coefficients computed using equation (10) are not independent any more due to the approximations involved in the equation itself. Their correlations though are minimal. For the efficient evaluation of (10) optimum procedures have been developed and can be found in Rapp (1986).

Spherical harmonic coefficients representing the square integrable analytical function $f(\phi,\lambda)$ can also be computed by a least squares adjustment of the sampled values of the function. Equations (1) or (8) can be used for that purpose as observation equations depending on whether the sampling is in a form of point values or area means. In a least squares solution individual data errors can be taken into account for weithting purposes, or all weights may be set equal to one. In the latter case and for point values, the formulation and the results are identical to the ones of equation (6) since the orthogonality principle is valid and so the normal matrix becomes diagonal. When mean values are used, with equal weights, the normal matrix is dominantly diagonal and the results and formulation are very similiar to the ones of equation (10) but not identical. In either formulation the coefficients can be considered as independent and insensitive to the maximum degree of solution. Furthermore, when the data sampling implies a Nyquist frequency that is greater that the maximum frequency existing in the function data (when the function is band limited), all coefficients corresponding to frequencies between these two frequencies will be effectively zero. Obviously, after the coefficients are estimated, a subset of them can be used to compute a long wavelength approximation of the function (i.e. a low pass filtering) or, equally well, high degree variations (i.e. a high pass filtering). Finally the spectrum of the function can be computed by computing the degree variances as follows

$$\sigma_\ell^2 = \sum_{m=0}^{\ell} (\bar{C}_{\ell m}^2 + \bar{S}_{\ell m}^2) \tag{11}$$

When the function $f(\phi,\lambda)$ is not globally defined on the sphere but only on a portion of it, then it is not analytical on the sphere anymore since its spatial derivatives are discontinuous on the boundaries. Consequently it cannot be expanded into a series of surface spherical harmonics that are orthogonal, since the orthogonality principle is not valid anymore. Ideally, such a function can be spectrally decomposed, if one is able to compute the eigenfunctions and eigenvalues of the function over the domain where the function is defined. When the boundaries though are variable, as it happens in most geodetic applications for which data are not sampled globally (e.g. satellite altimetry) such a computation becomes nearly impossible. Furthermore, it has been traditional for geodesists to work with spherical harmonics since all the geodetic quantities of interest can be analysed into spherical harmonics.

Based on the above, one can expand $f(\phi,\lambda)$ into spherical harmonics, only with the understanding that such an expansion is nothing else than a polynomial fit, and that the only property of the harmonic coefficients is that they reproduce the function. To compute the coefficients of equations (1) or (9), depending on the nature of the available sampling, one cannot use equations (6) or (10) anymore. The only way these coefficients can be computed is through a least squares fit of the data using (1) or (9) as observation equations. A similiar situation arises when a function that is globally defined on the sphere is not globally sampled (e.g. terrestrial gravity anomalies).

There are several issues that need to be addressed during such a computation. First of all and most important, the coefficients themselves are not independent but correlated. This is expected since the normal matrix is not diagonal or even dominantly diagonal. Therefore, no long wavelength approximation to the function can be computed and equation (11) cannot really be used to compute the spectrum. Furthermore, the Nyquist frequency cannot be defined, based on the sampling of the function, in the sense $\ell_{max} = \pi/\Delta\lambda$ since the discontinuities at the boundaries introduce artificial energies at frequencies that are functions of the variability of the boundaries. This effect is the so called Gibbs phenomenon, which is very well known in the theory of Fourier series. So even if the function is band limited with a maximum frequency ℓ less than the maximum frequency ℓ_{max} that is implied by the data, if one tries to solve up to ℓ_{max} then the coefficients between ℓ and ℓ_{max} will not be zero but will have substantial magnitudes. Furthermore, considerable folding of these frequencies occurs so that to also affect the coefficients of degrees lower than ℓ. As it will be seen later on, the greater the maximum degree of the solution is, the larger the folding and therefore the larger the coefficients become. Of course the correlations between these coefficients aldo increase in such a way that, when they are used to reproduce the function, they provide a perfect fit.

From this discussion it is obvious that the computed coefficients are not unique but they depend on the maximum degree of solution. So, if a band limited function is analyzed, or smoothing has been applied to the data, it is important that some prior information about the effective minimum wavelengths (and therefore the maximum degree of solution) be available.

In analyzing an incomplete data set on the sphere, either because of definition or because of incomplete sampling, the geometry of the data distribution is very important, and is basically the factor for the deviation of the harmonics from orthogonality. As will be seen later on, a very small change in the data distribution can result in substantial changes in the computed coefficients even when the same set of coefficients is solved for (i.e. same maximum degree of solution). Unfortunately there is no known method that can estimate what the effect of a gap in the data on the computed coefficients can be. The only information that can be obtained is how a particular gap of data affects the correlations of the coefficients. This can be done by simply computing the normal matrix for the gap. Doing that repeatedly, for several gaps, one can possibly identify areas to which a solution is very sensitive and areas that contribute a little to the solution.

Determination of the SST Spherical Harmonic Coefficients.

In determining the Levitus SST harmonic coefficients Engelis (1985) has adopted the definition that the SST is a global function that takes values both on land and oceans, with the values on land being identically equal to zero. In that definition it was required that the mean value of the SST is zero as sampled globally in the oceanic areas of the world. The SST estimates that were used in such an ivestigation were the original Levitus values that correspond to SET1. These estimates were considered to be point values. Then, spherical harmonic coefficients were computed using equation (6). The same analysis was repeated in Rapp (1985) where though SET3 was used and the values were considered to be mean values. Equation (10) was then used to compute the SST coefficients. In both solutions the results were practically insensitive to the maximum degree of expansion. The difference between the two sets of coefficients was found to be on the order of millimeters for most of the coefficients with the exception of the first and second degree terms that differ by a couple of centimeters. Furthermore the cumulative power computed by either solution (up to ℓ_{max} = 36) was on the order of 40 cm which is substantially lower than the RMS values of the Levitus sets SET1 and SET3 that are on the order of 80.3 and 62.4 cm respectively. This substantial reduction was due to the smoothing of the derived fields that were forced to also fit the zeroes on land. For the same reason the RMS fit of these fields to the data was only on the order of 10-20 cm.

The same type of solution is presently repeated by applying a least squares fit to the Levitus data set augmented by zeroes on land. Several solutions up to different maximum harmonic degrees (6,10,20,36) have been made for all the four sets of SST described previously. In all the solutions equal weights were used. Equation (9) was used to form the observation equations. The conclusions that can be drawn from the adjustment results are identical to the ones drawn from the quadratures solutions described above. Again there is no variation of the coefficients among solutions with different ℓ_{max} for any of the data sets. Furthermore solutions using SET1 and SET2 as well as solutions using SET3 and SET4 are almost identical (differences on the order of millimeters). This level of differences was also found in comparing these solutions with the corresponding quadratures solutions. In any of these least squares fits the standard deviation of each individual coefficient is 2 cm while the correlations were found to be indeed negligible. Any solution up to degree 36 has an RMS power of 40 cm and an RMS fit to the Levitus data of 11 cm. Solutions up to degree 10, on the other hand, have an RMS fit of 20 cm and an RMS power of 37 cm.

Solutions for the SST harmonic coefficients have also been made by considering the SST to be defined only in the oceanic regions. In order to examine the sensitivity of the coefficients to the choice of the maximum harmonic degree and the geometry of the data distribution, solutions up to 8, 10, 12, 20 and 36 were made using all the four data sets. In establishing which set of coefficients is the best, a combination of the following criteria was used.

1. RMS fit to the Levitus data to be small.

2. RMS value implied by the cumulative degree variances to be close to the RMS value of the Levitus SST.

3. Zero degree coefficient to be very close to zero.

4. Estimated standard deviations of the coefficients to be small.

5. The condition number of the normal matrix to be small.

This set of criteria was devised as an empirical alternative way to choose the set with the smallest possible correlations since examining the normal matrices for all these sets was found to be impractical. In particular, the combination of the first and second criteria can give a first indication about the level of correlations of coefficients. The third criterion can support the first two. Indeed, by definition the zero degree term has to be zero, so any deviation from zero indicates the level of correlation of the zero degree term with the other coefficients. Finally the fourth criterion is useless by itself but if combined with the other criteria and particularly with the last one it can give one further input on the choice of the best set of coefficients.

Examining all the different solutions it was found that the solution up to harmonic degree 10 using SET3 performs the best, since it better satisfies all the criteria. More specifically the RMS discrepancy from SET3 is 7 cm, the RMS value implied by the degree variances is 62.8 cm (closest to the RMS value of SST which is 62.4 cm) and the zero degree term is -4 cm. The standard deviations of the coefficients (not scaled by the aposteriori variance of unit weight) range from 2 to 30 cm and the condition number is on the order of 5000. These accuracy estimates are poorer than the corresponding ones from solutions to a smaller maximum degree (i.e. 8) but only marginally. On the contrary they are much smaller than the standard deviations of higher degree solutions.

The performance of the solutions up to harmonic degree 10 using SET1 and SET2 is almost comparable. Indeed the first two criteria are satisfied since the RMS fits to the corresponding data are on the order of 9 cm and their cumulative power reaches 80 cm. The other criteria though are not satisfied as well since the zero degree terms of the two solutions are -25 cm and -20 cm respectively and the standard deviations of all the coefficients are somehow larger than those of the SET3 solution. The reason for this is that the systematically lower values at latitudes greater than 70° introduce a step discontinuity in the data. This discontinuity affects primarily the zonals and particularly the second degree zonal that is on the order of -55 cm as compared to -28 cm of the SET3 solution. The coefficients computed from SET4 perform the worse. Indeed some of these coefficients have magnitudes greater than 50 cm giving a cumulative power in excess of 1 meter. Still, the RMS discrepancy from the SET4 data set is 7 cm. This indicates that there are high correlations between the coefficients and that the solution is extremely sensitive to the geometry of the data distribution. Indeed a small change of SET3 (removal of 282 values in the Mediterrannean) makes the solution (i.e. SET4) completely unreliable.

For all the data sets, any solution with a maximum harmonic degree greater than 10 gives coefficients that are completely unreliable, since they reach magnitudes close to 1 meter and standard deviations of several meters, while the condition numbers increase dramatically. In all these solutions the RMS discrepancies from the Levitus data reduce with increasing ℓ_{max} to reach a value of 8 mm for a solution up to degree 36. For this particular solution the cumulative power is on the order of 8 meters, the standard deviations are on the order of 10-15 meters and the condition number is on the order of 10^8. Even a solution up to degree 12, although it gives an RMS fit of 6 cm to the Levitus data, it has a cumulative power of 1.35 m and individual standard deviations on the order of 50 cm.

The coefficients of the ocean solution up to degree 10 and based on SET3 are shown in Table 1 together with their by degree and cumulative amplitudes. Additional sets that are given for comparison are the ones of the global solution up to degree 10 (Table 2), and the ocean solution up to degree 36 (Table 3), always based on SET3. All these Tables can be found in Appendix B. Comparing the degree amplitudes between the oceanic and global solutions to degree 10 we observe a higher energy in the oceanic solution, which was expected, but also a similar decay in energy with increasing degree. Such a behavior and the fact that many of the SST features above degree 10 have been filtered out by Levitus suggest that the spectral content of the SST could be approximated by the estimates of Table 1 reasonably well.

The three sets of coefficients given in the Tables as well as the coefficients of the global SET3 solution to ℓ_{max} = 36, have been used to generate SST estimates on a 1°x1° grid. Contour maps of these estimates based on a 5°x5° grid were then generated and are shown in Figures 6-9 of Appendix A, while a contour map of the Levitus set (SET3) also based on a 5°x5° is shown in Figure 5. Examining Figures 6 and 7, that portray the SST from the ocean and global solutions up to degree 10 respectively, it is seen that the primary difference is that the former is able to show the broad features of the Gulf stream in a much better way than the latter. Examining Figures 8 and 9 that portray the SST features to degree 36 we can see that all features of the SST are almost identical and have an excellent agreement to the ones of Figure 5. In a more detailed visual examination the SST in the Gulf stream region from the SET3 data set and the four solutions has been plotted and is shown in Figures 10 through 14 of Appendix. Now one can clearly see the inadequacy of the global solutions to reproduce the SST features faithfully, although the global solution to degree 36 provides a good qualitative approximation. The ocean only solution up to degree 36 is identical to the original data.

In order to create Figures 8 and 13 the SST values on land were replaced by the corresponding values of the global solution to ℓ_{max} = 36, since the land values of the ocean solution to ℓ_{max} = 36 are on the order of 20 meters and no contouring could be made. The same values have been used to plot the Levitus maps in Figures 5 and 10. Such a substitution may have created small distortions close to the coastlines which though are expected to be minor. In any case the values on land and close to the coastlines are meaningless.

Conclusions.

From the analysis made so far it is understood that the determination of spherical harmonics from the oceanic Levitus data set is very sensitive to a number of factors. It is seen that high correlations exist between the recovered coefficients that makes them completely unreliable, although they provide a good fit to the data. The best set of coefficients among solutions has been identified (SET3 ocean solution to ℓ_{max} = 10) based on certain criteria. Additional solutions which are as good (SET1 and SET2 ocean solutions) have been found but have not been chosen because they also reflect the data in the north polar region and therefore they do not provide an equally good fit to the ocean areas also attained by altimetry, as the SET3 solution does. The experience obtained from this investigation indicates that although the chosen solution is the best among solutions that have been attempted it may very well turn to be suboptimum. As a matter of fact, it may be possible that some other slightly different configuration of the data and/or solution may very well give better results. Based on the above, one could even be inclined to use the coefficients computed from the global solutions since such solutions although, not giving very good fits, have very well established properties.

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Appendix A

Sea Surface Topography Maps

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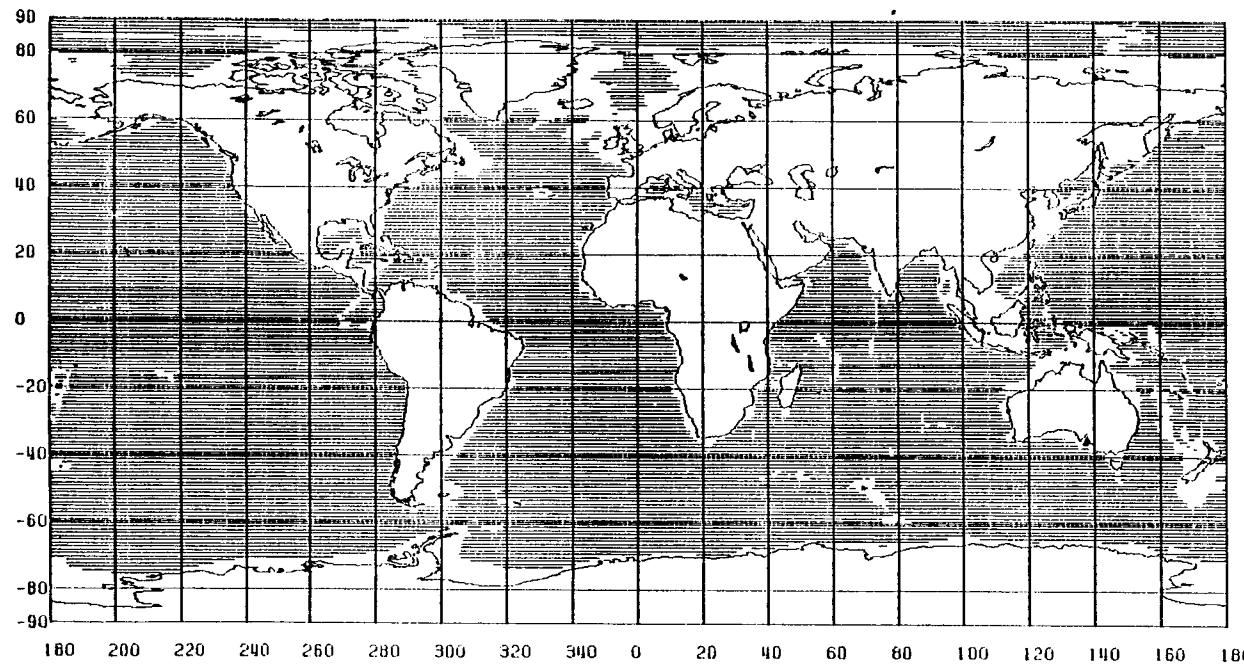

Figure 1. Spatial Distribution of 33856 1°x1° Original Levitus Values (SET1)

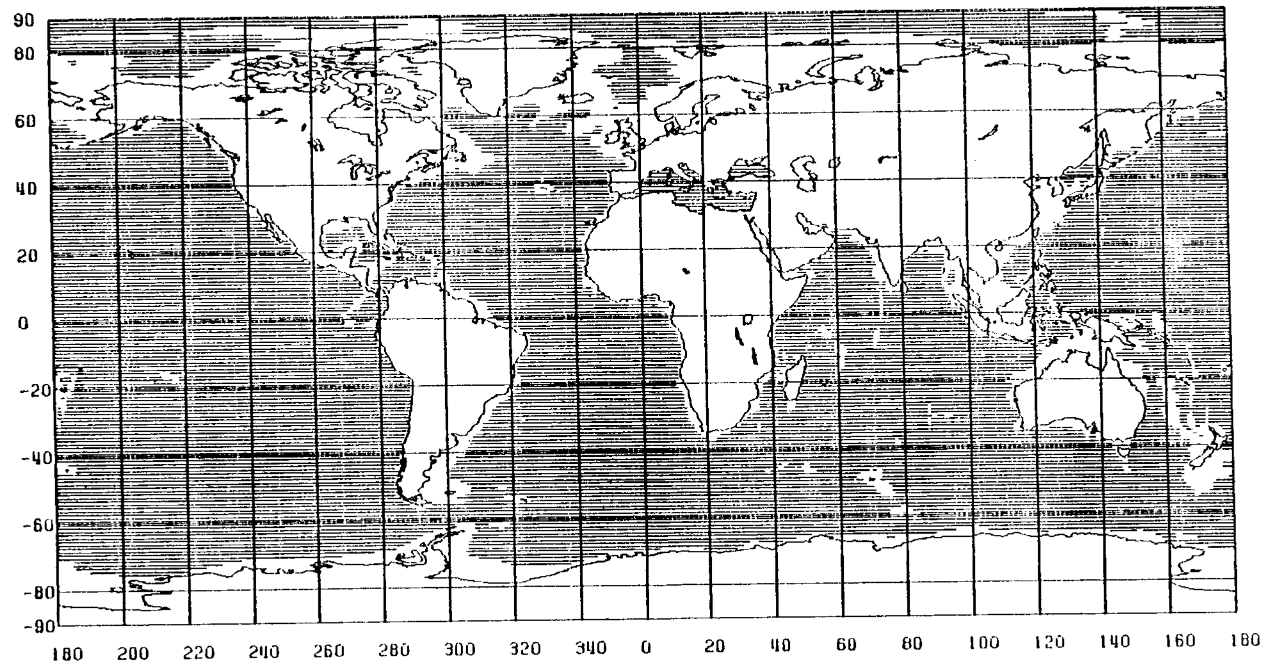

Figure 2. Spatial Distribution of SET2 34056 1°x1° SST Values

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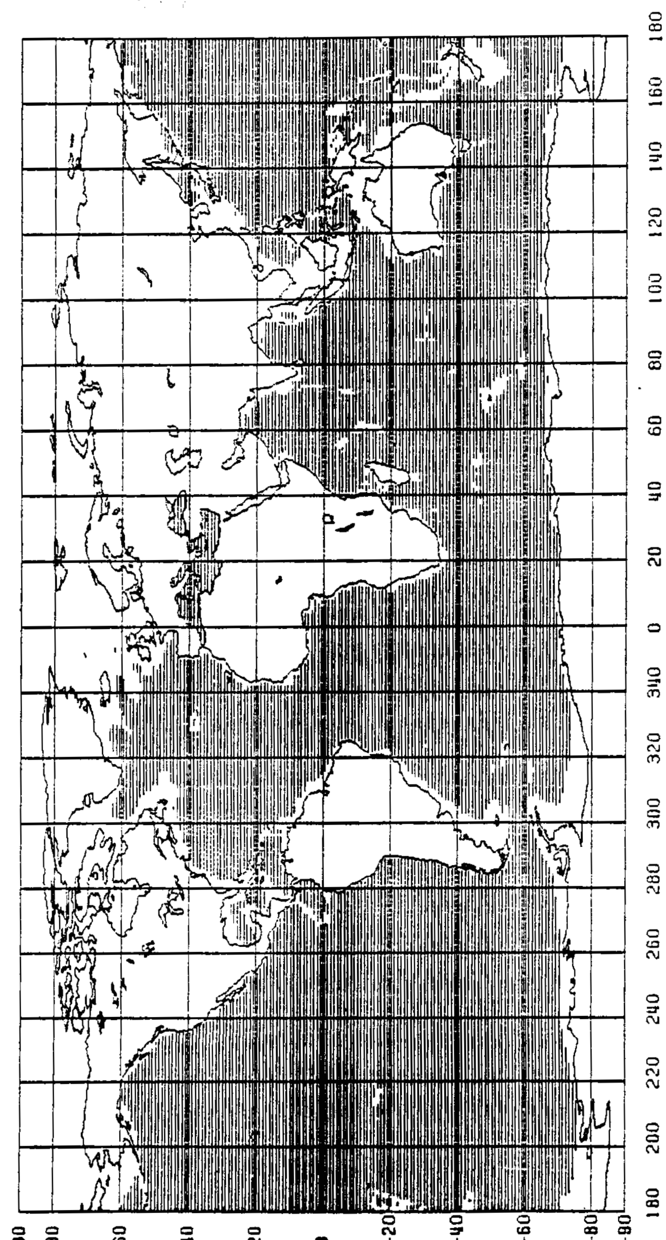

Figure 3. Spatial Distribution of SET3 30922 1°x1° SST Values

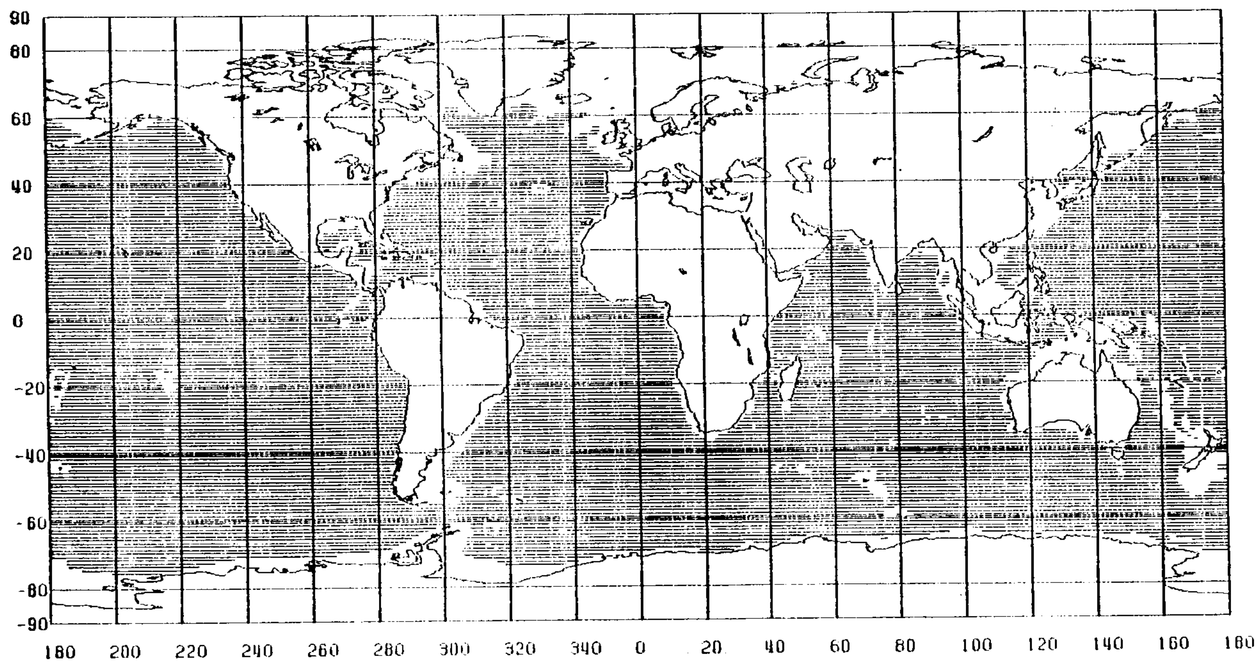

Figure 4. Spatial Distribution of SET4 30640 1°x1° SST Values

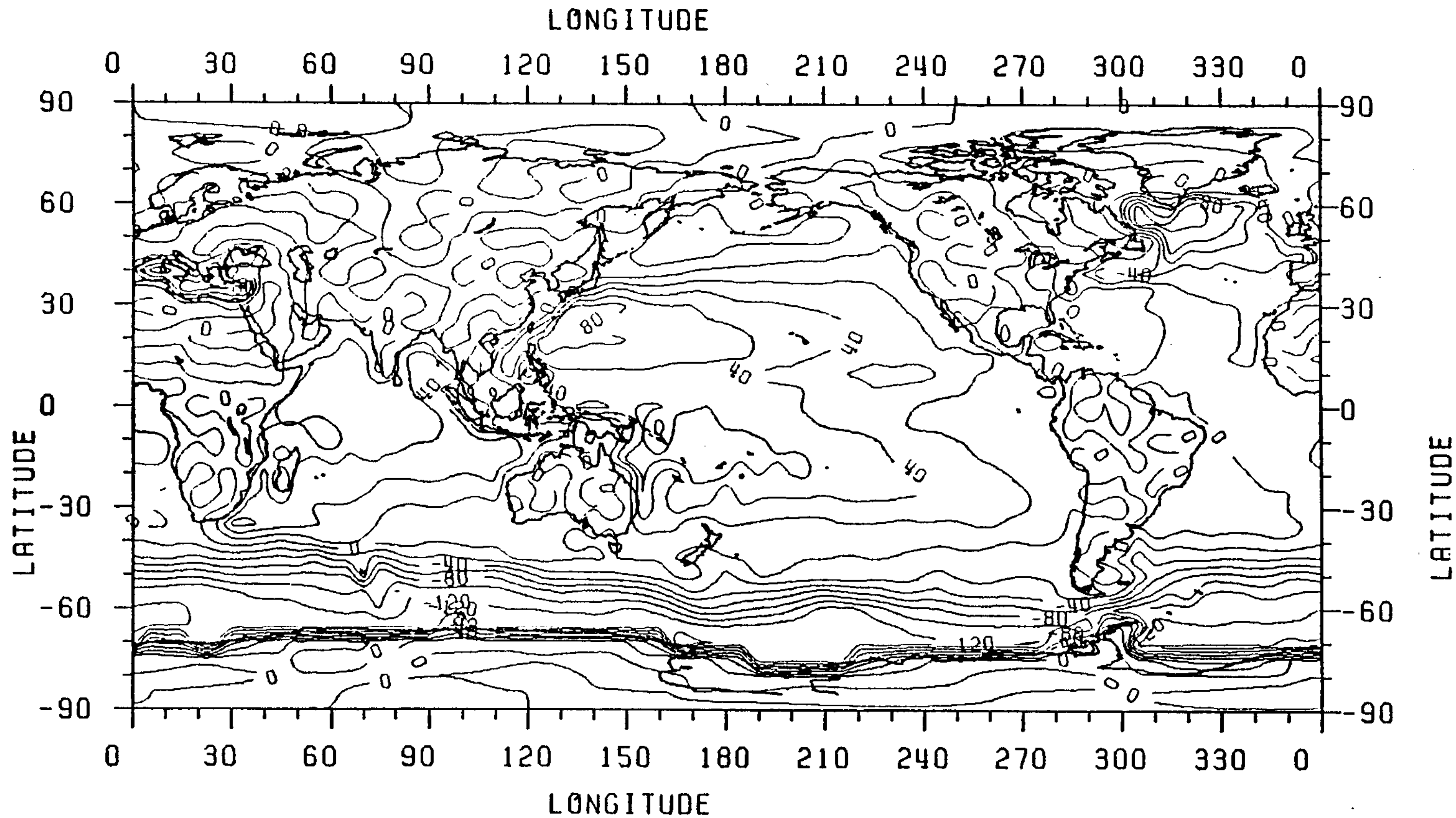

Figure 5. Levitus SST Estimates (SET3), Based on a 5°x5° Grid. C.I. = 20 cm.

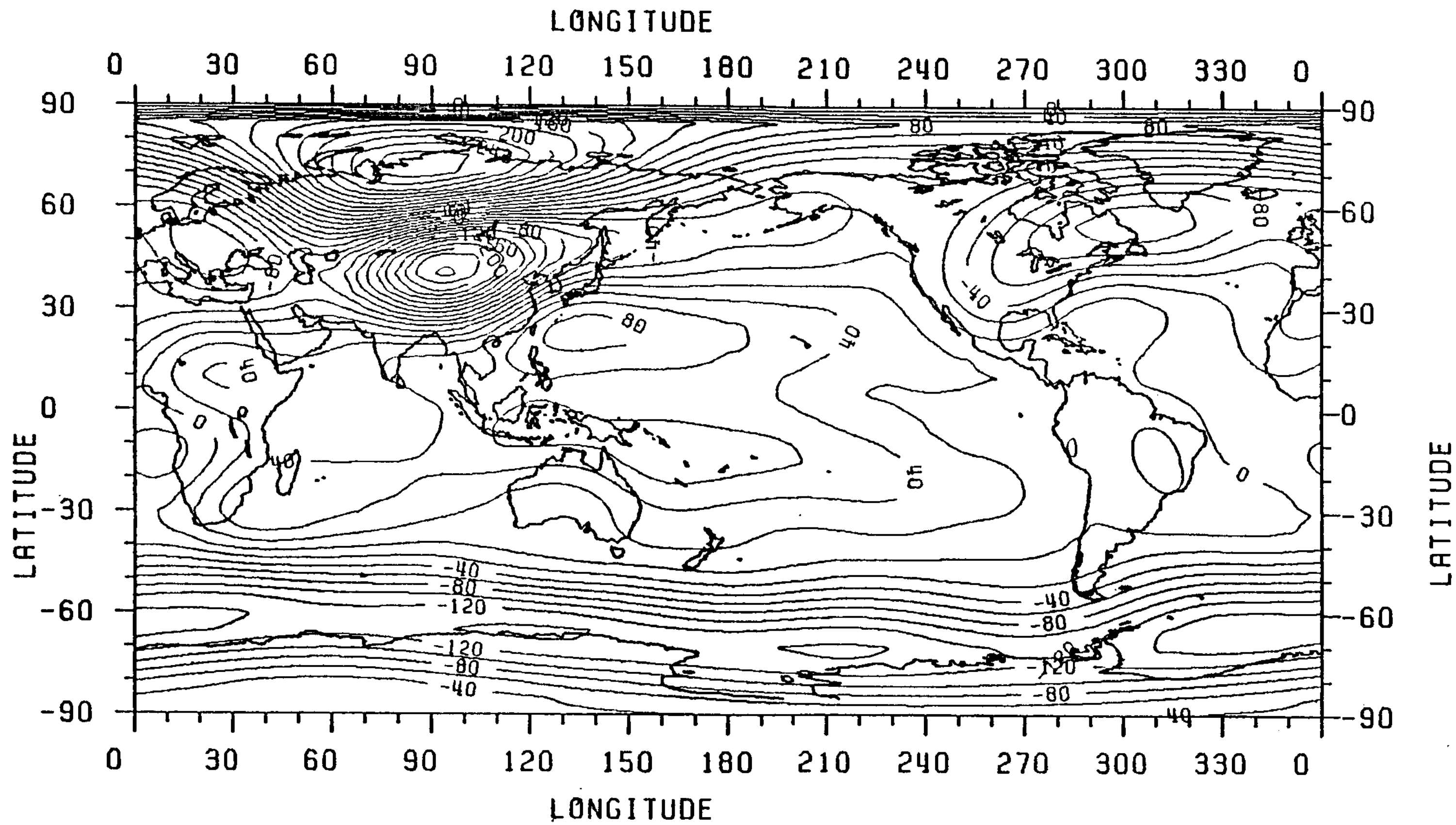

Figure 6. SST Estimates from the Ocean Analysis of Levitus Data (SET3) up to Harmonic Degree 10, Based on a 5°x5° Grid. C.I. = 20 cm.

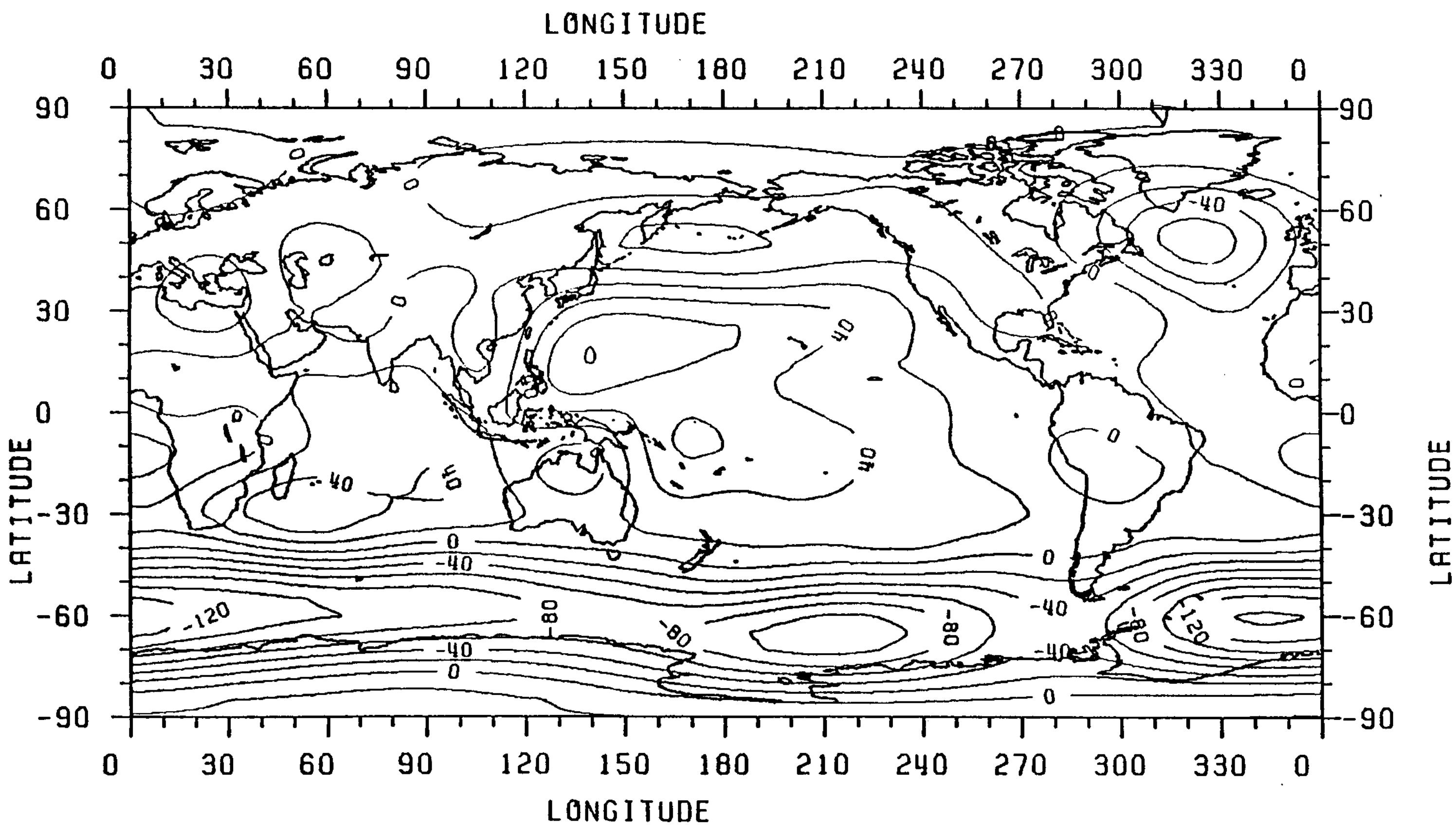

Figure 7. SST Estimates from the Global Analysis of Levitus Data (SET3) up to Harmonic Degree 10, Based on a 5°x5° Grid. C.I. = 20 cm.

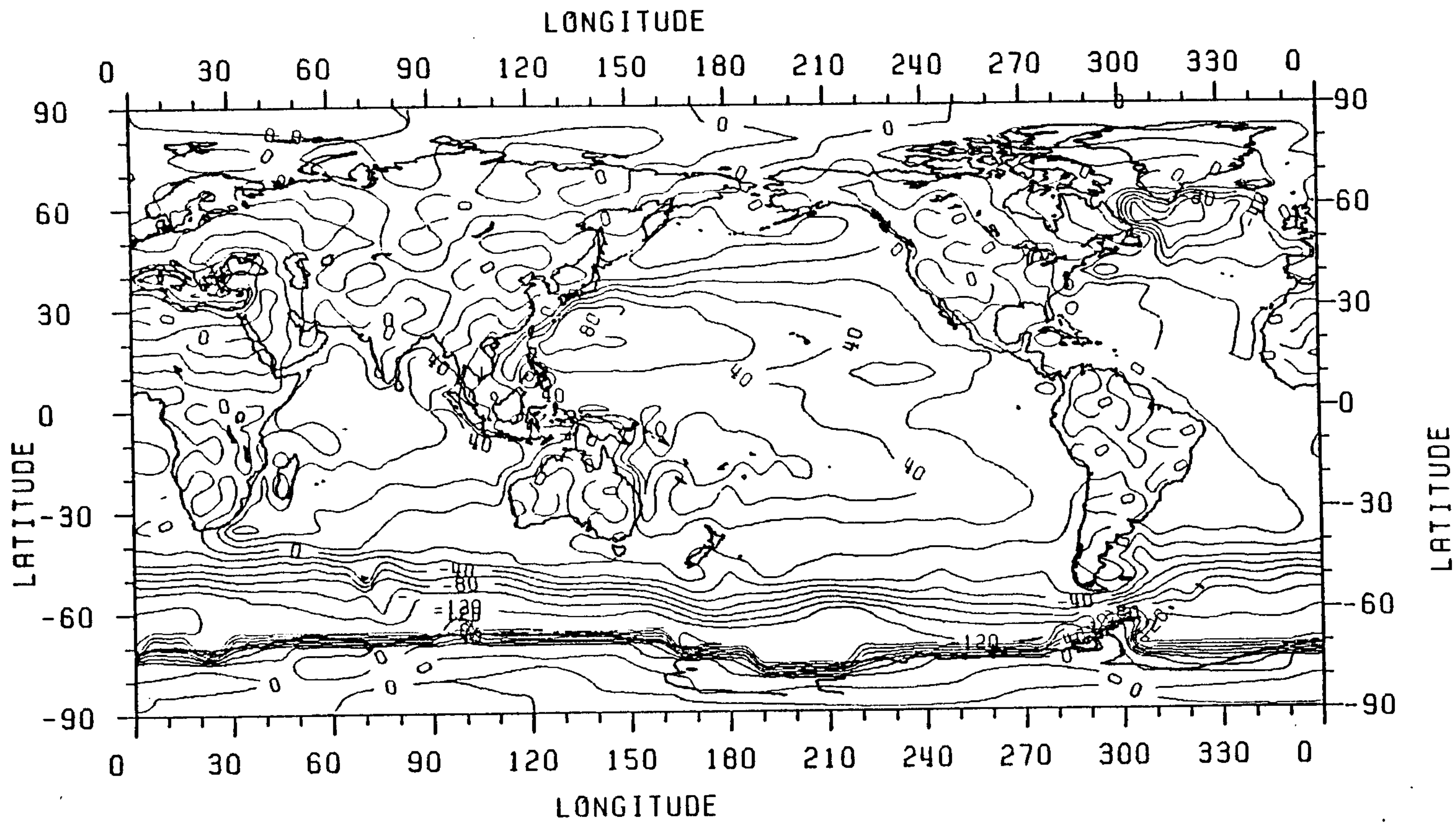

Figure 8. SST Estimates from the Ocean Analysis of Levitus Data (SET3) up to Harmonic Degree 36, Based on a 5°x5° Grid. C.I. = 20 cm.

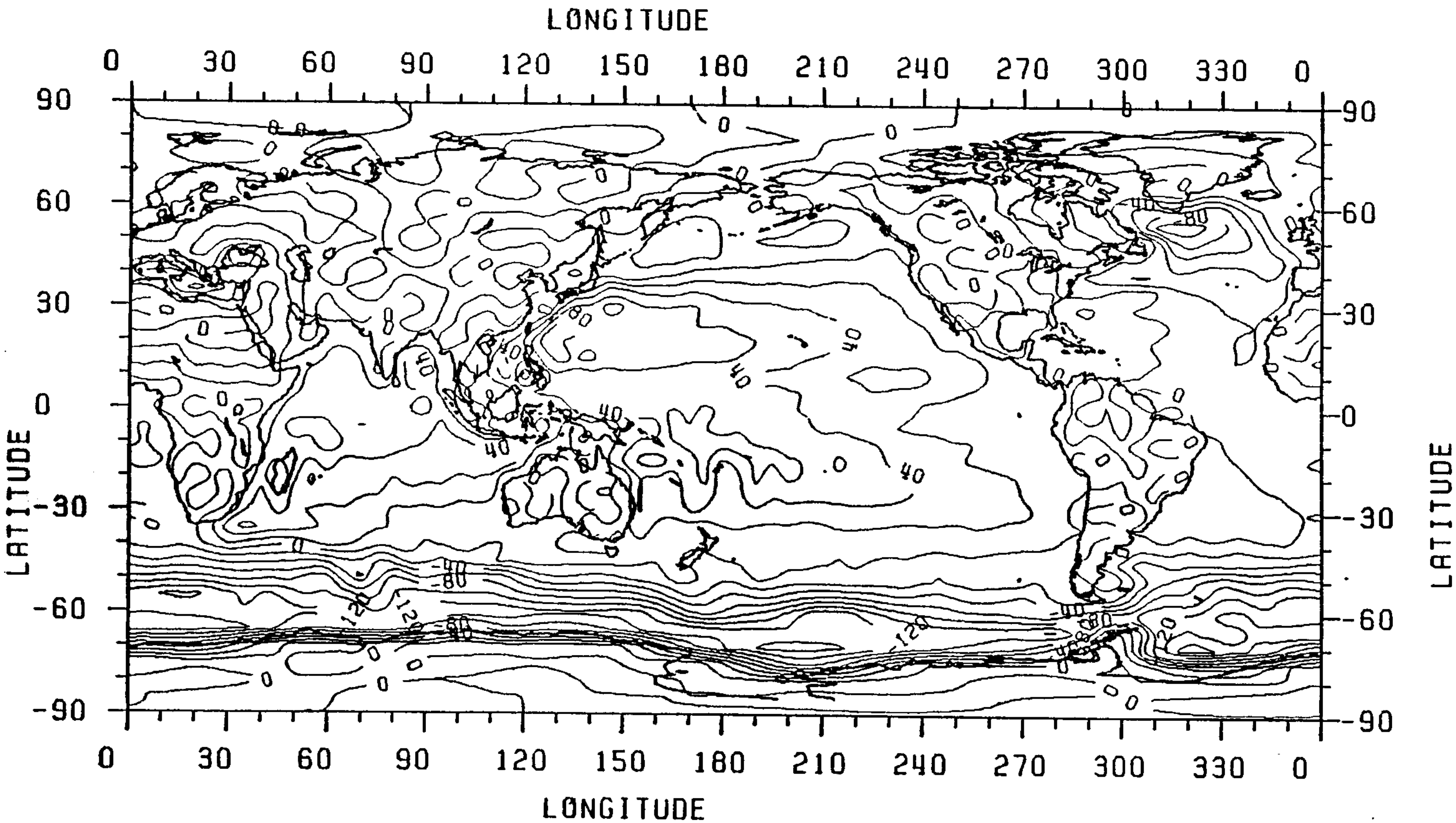

Figure 9. SST Estimates from the Global Analysis of Levitus Data (SET3) up to Degree 36, Based on a 5°x5° Grid. C.I. = 20 cm.

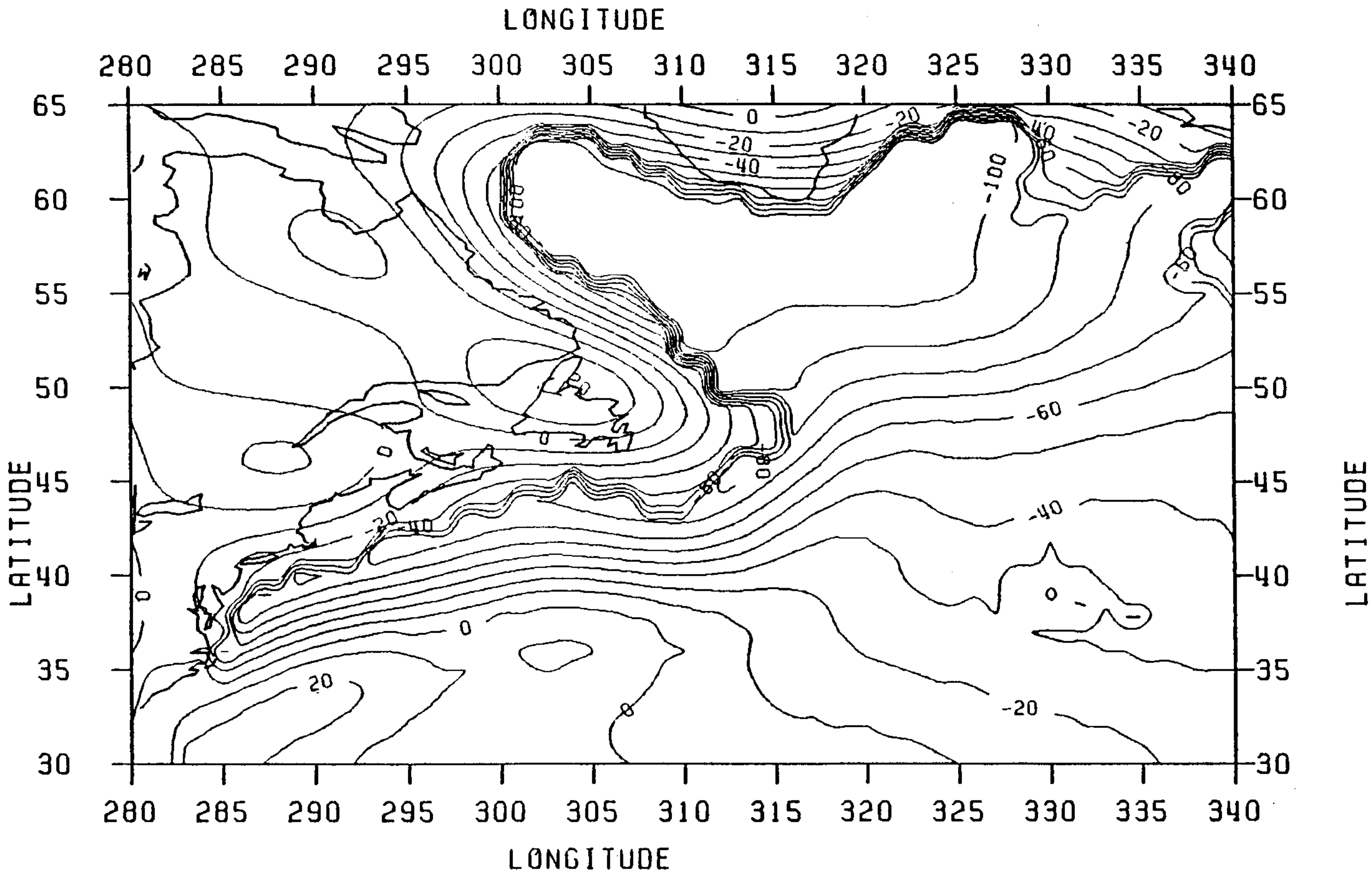

Figure 10. Levitus SST Estimates (SET3) in the Gulf Stream Region, Based on a 1°x1° Grid. C.I. = 10 cm.

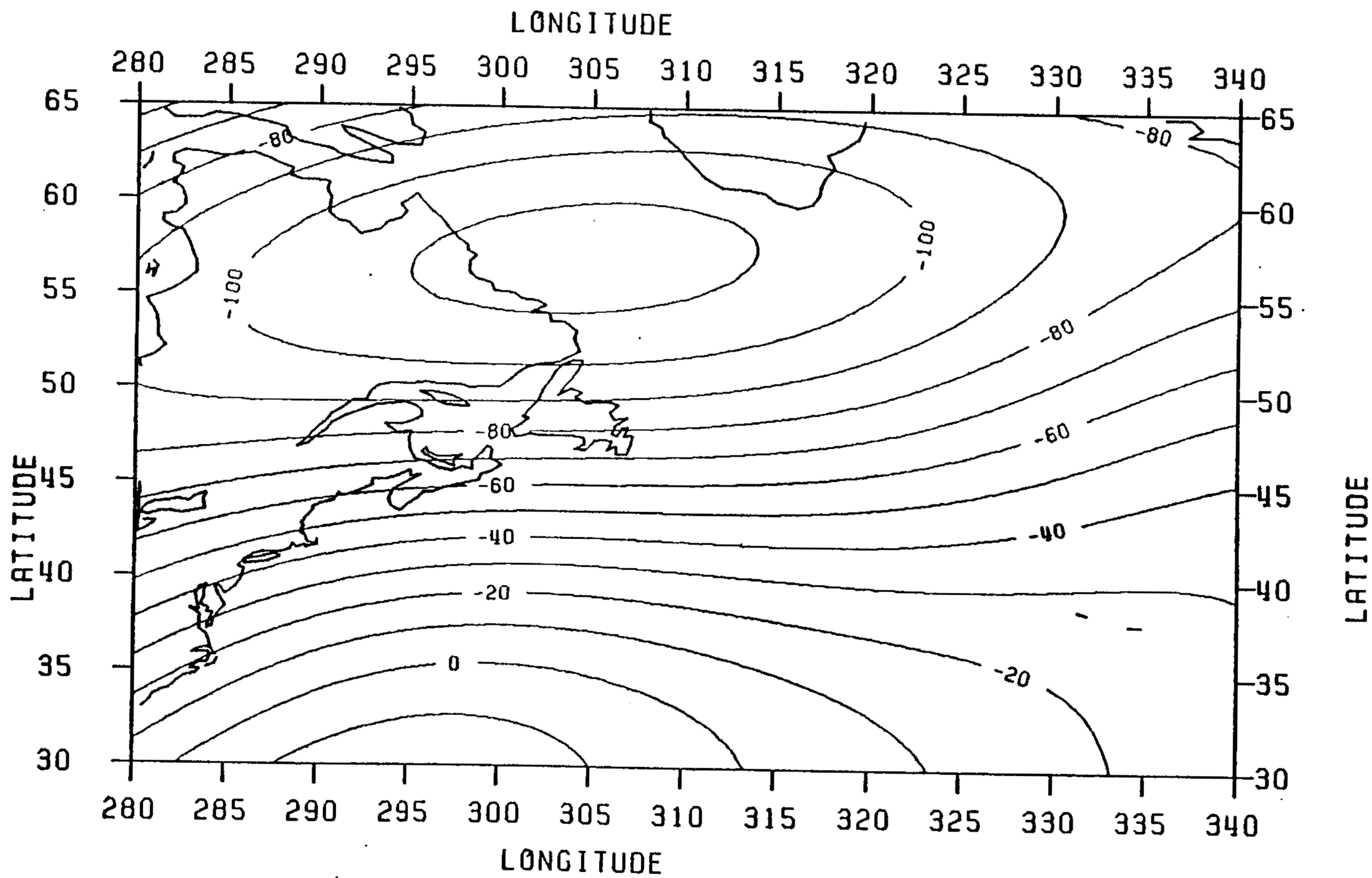

Figure 11. SST Estimates from the Ocean Analysis of Levitus Data (SET3) up to Harmonic Degree 10 in the Gulf Stream Region, Based on a 1°x1° Grid. C.I. = 10 cm.

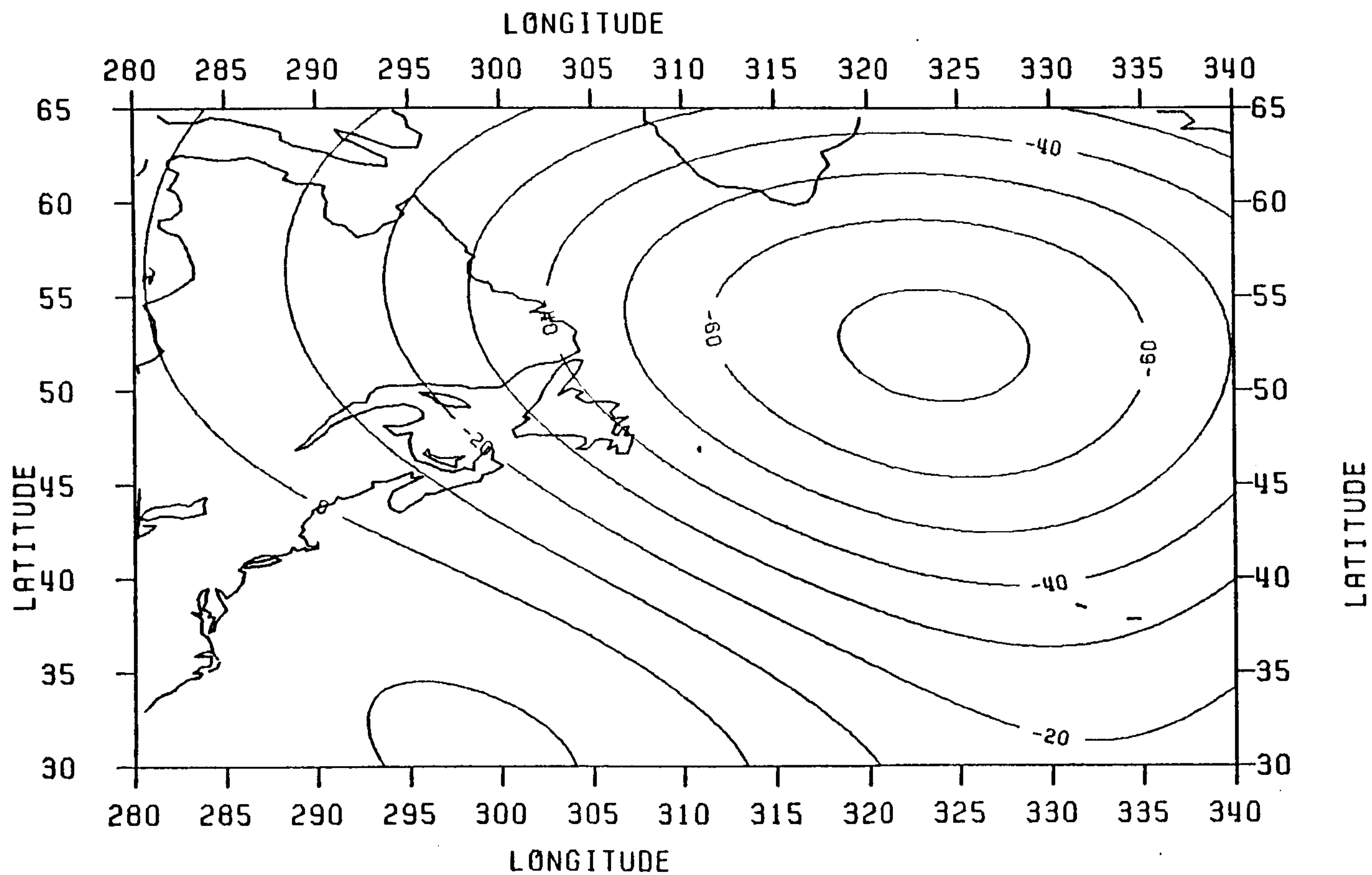

Figure 12. SST Estimates from the Global Analysis of Levitus Data (SET3) up to Harmonic Degree 10 in the Gulf Stream Region, Based on a 1°x1° Grid. C.I. = 10 cm.

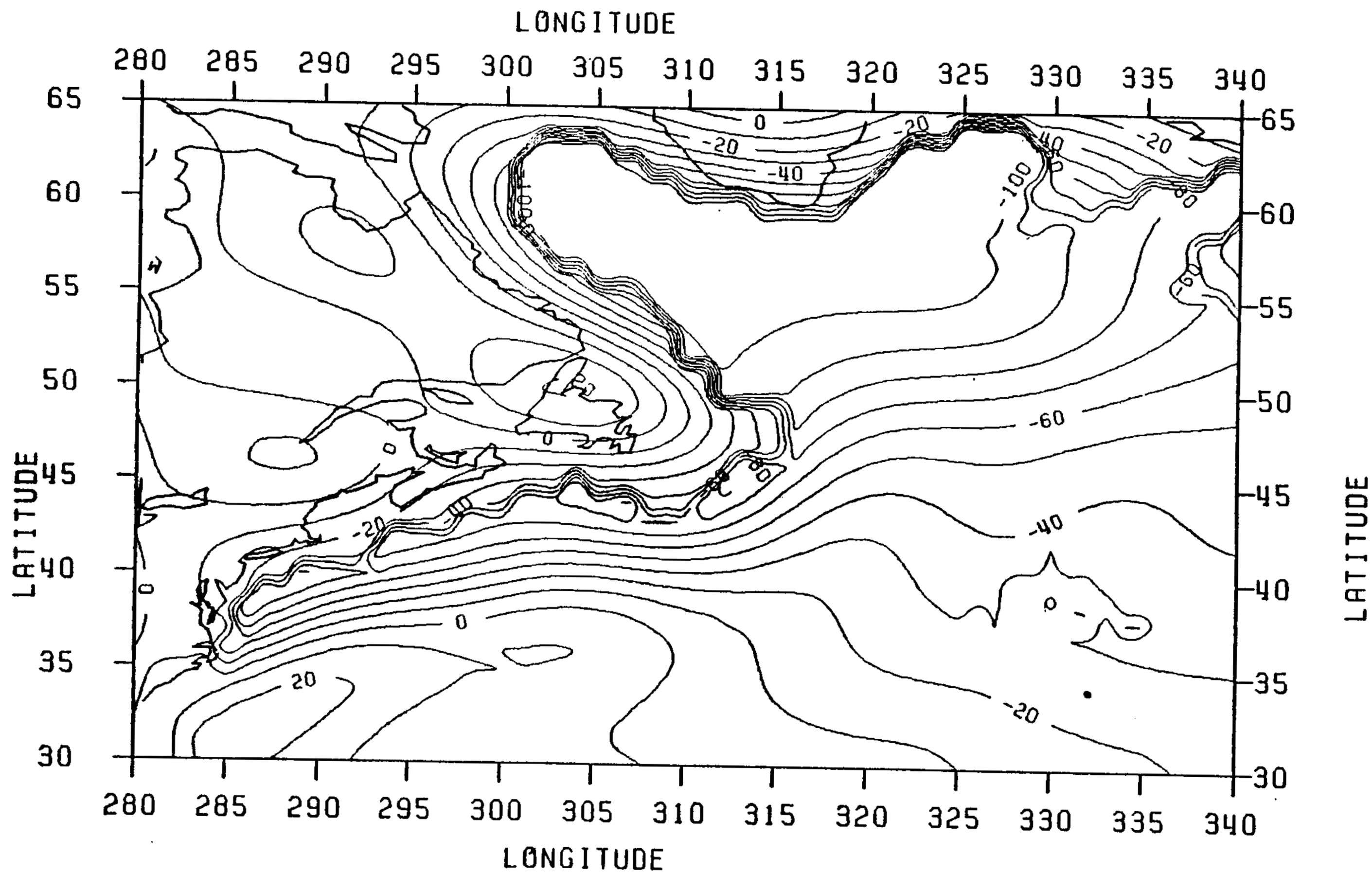

Figure 13. SST Estimates from the Ocean Analysis of Levitus Data (SET3) up to Harmonic Degree 36 in the Gulf Stream Region, Based on a 1°x1° Grid. C.I. = 10 cm.

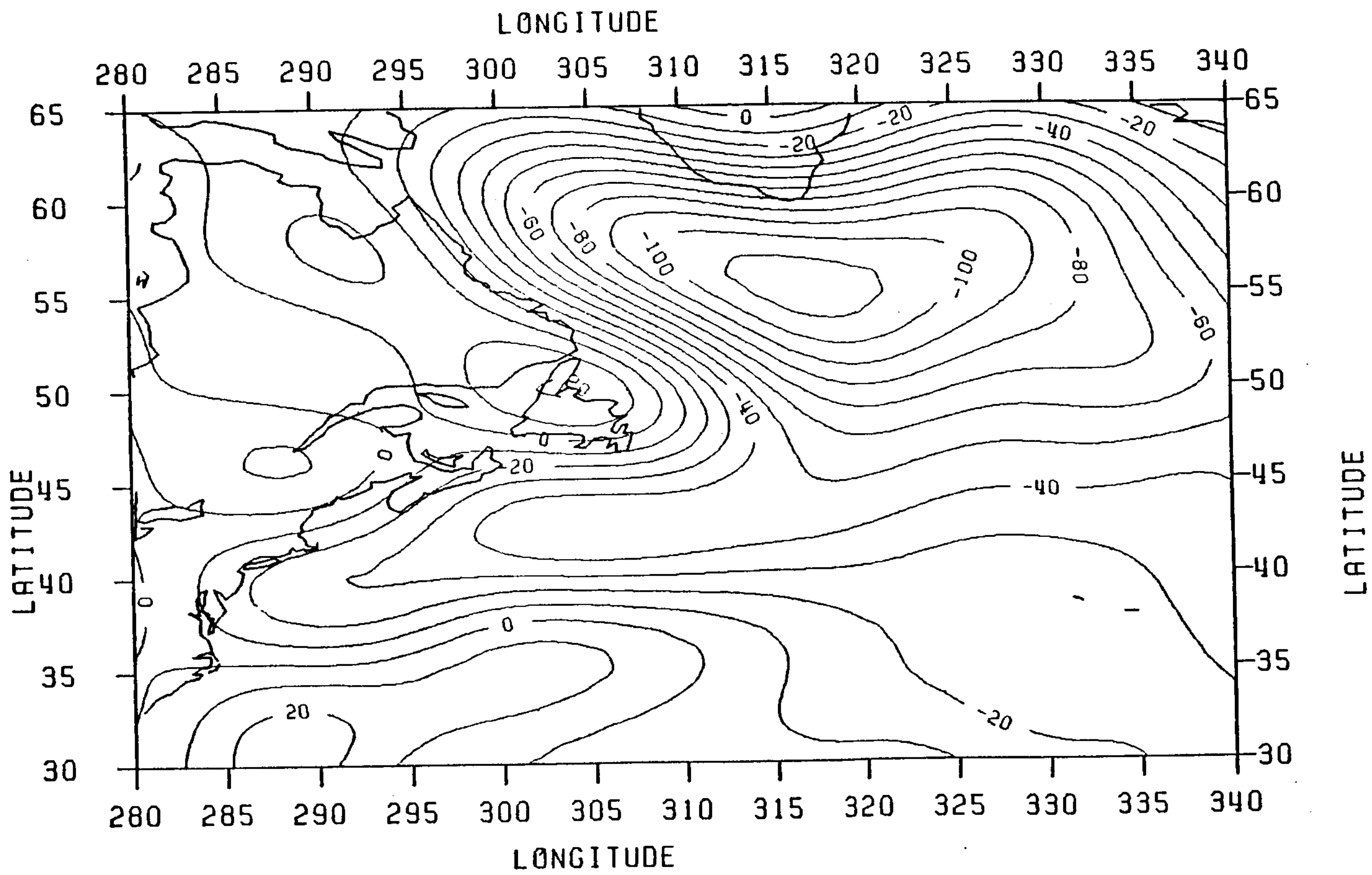

Figure 14. SST Estimates from the Global Analysis of the Levitus Data (SET3) up to Harmonic Degree 36 in the Gulf Stream Region, Based on a 1°x1° Grid. C.I. = 10 cm.

Appendix B

Spherical Harmonic Coefficients from Different Solutions of the Levitus SST Estimates. Units are in meters.

TABLE 1

OCEAN SOLUTION BASED ON LEVITUS SST (SET3)

MAXIMUM DEGREE OF EXPANSION = 10
NUMBER OF UNKNOWNS = 121
RECIPROCAL CONDITION NUMBER = 0.593E-05
DETERMINANT = 1.633*10.0**505

NUMBER OF SST VALUES USED = 30922
DEGREES OF FREEDOM = 30801
RMS FIT TO LEVITUS DATA = 0.07M

N	M	CNM	SNM	SN	SN(CUM)
0	0	-0.0406	0.0000	0.0406	0.0406
1	0	0.1297	0.0000	0.2220	0.2257
1	1	-0.1766	0.0360		
2	0	-0.2803	0.0000	0.2854	0.3639
2	1	-0.0452	0.0099		
2	2	0.0261	0.0087		
3	0	0.2663	0.0000	0.2767	0.4572
3	1	-0.0194	-0.0111		
3	2	0.0507	-0.0141		
3	3	-0.0391	0.0295		
4	0	0.1467	0.0000	0.1922	0.4959
4	1	0.0123	0.1053		
4	2	0.0201	0.0368		
4	3	0.0010	0.0239		
4	4	-0.0432	-0.0005		
5	0	0.1242	0.0000	0.2064	0.5372
5	1	0.0008	0.1494		
5	2	-0.0129	0.0470		
5	3	-0.0268	0.0221		
5	4	-0.0337	-0.0030		
5	5	-0.0114	-0.0029		
6	0	0.1536	0.0000	0.2294	0.5841
6	1	-0.0083	0.1252		
6	2	-0.0824	0.0286		
6	3	-0.0199	-0.0286		
6	4	-0.0417	-0.0416		
6	5	0.0283	-0.0143		
6	6	0.0012	0.0017		
7	0	-0.0754	0.0000	0.1685	0.6080
7	1	0.0094	0.1012		
7	2	-0.0807	0.0223		
7	3	-0.0045	-0.0108		
7	4	-0.0366	-0.0472		
7	5	0.0217	-0.0257		
7	6	0.0183	0.0138		
7	7	-0.0037	0.0006		
8	0	0.0057	0.0000	0.1098	0.6178
8	1	-0.0582	-0.0090		
8	2	-0.0756	0.0102		

8	3	0.0082	-0.0200		
8	4	0.0092	-0.0281		
8	5	0.0174	-0.0210		
8	6	0.0195	-0.0015		
8	7	-0.0142	0.0034		
8	8	-0.0027	-0.0063		
9	0	-0.0651	0.0000	0.0999	0.6258
9	1	0.0248	0.0084		
9	2	-0.0326	-0.0399		
9	3	-0.0103	-0.0144		
9	4	0.0151	-0.0237		
9	5	0.0138	-0.0189		
9	6	0.0238	0.0021		
9	7	-0.0079	0.0026		
9	8	-0.0080	0.0048		
9	9	0.0017	-0.0050		
10	0	0.0112	0.0000	0.0602	0.6287
10	1	0.0217	-0.0356		
10	2	-0.0064	-0.0154		
10	3	0.0018	-0.0154		
10	4	0.0225	-0.0140		
10	5	-0.0007	0.0006		
10	6	0.0205	0.0014		
10	7	0.0029	0.0029		
10	8	-0.0046	-0.0004		
10	9	-0.0018	0.0080		
10	10	-0.0018	-0.0017		

TABLE 2

GLOBAL SOLUTION BASED ON LEVITUS SST (SET3)

MAXIMUM DEGREE OF EXPANSION = 10
NUMBER OF UNKNOWNS = 121
RECIPROCAL CONDITION NUMBER = 0.948E-01
DETERMINANT = 3.353*10.0**575

NUMBER OF SST VALUES USED = 64800
DEGREES OF FREEDOM = 64679
RMS FIT TO LEVITUS DATA = 0.19M

N	M	CNM	SNM	SN	SN(CUM)
0	0	0.0014	0.0000	0.0014	0.0014
1	0	0.1126	0.0000	0.1827	0.1827
1	1	-0.1419	0.0239		
2	0	-0.2195	0.0000	0.2224	0.2878
2	1	-0.0120	0.0184		
2	2	0.0052	0.0275		
3	0	0.0812	0.0000	0.0974	0.3039
3	1	0.0107	-0.0169		
3	2	0.0224	-0.0205		
3	3	-0.0397	0.0020		
4	0	0.0338	0.0000	0.0692	0.3116
4	1	0.0363	0.0092		
4	2	-0.0328	0.0084		
4	3	0.0159	0.0234		
4	4	0.0019	-0.0173		
5	0	-0.0402	0.0000	0.0673	0.3188
5	1	0.0021	0.0153		
5	2	-0.0089	0.0456		
5	3	-0.0037	0.0202		
5	4	-0.0057	0.0047		
5	5	0.0059	0.0033		
6	0	0.1220	0.0000	0.1333	0.3456
6	1	-0.0078	-0.0145		
6	2	-0.0303	-0.0039		
6	3	0.0092	-0.0252		
6	4	-0.0023	0.0032		
6	5	0.0307	0.0000		
6	6	0.0014	0.0022		
7	0	-0.0889	0.0000	0.1002	0.3598
7	1	0.0095	0.0185		
7	2	0.0242	0.0037		
7	3	0.0005	0.0228		
7	4	0.0123	0.0028		
7	5	0.0120	-0.0065		
7	6	0.0068	0.0139		
7	7	0.0015	-0.0010		
8	0	0.0318	0.0000	0.0710	0.3668
8	1	-0.0372	-0.0333		
8	2	-0.0021	-0.0080		

8	3	0.0227	0.0112		
8	4	0.0170	0.0133		
8	5	-0.0046	0.0023		
8	6	0.0065	-0.0112		
8	7	-0.0017	-0.0120		
8	8	0.0046	-0.0010		
9	0	-0.0127	0.0000	0.0431	0.3693
9	1	0.0163	0.0036		
9	2	0.0237	-0.0068		
9	3	-0.0031	0.0162		
9	4	-0.0008	-0.0015		
9	5	0.0127	-0.0012		
9	6	-0.0004	-0.0107		
9	7	0.0070	-0.0063		
9	8	0.0104	-0.0006		
9	9	0.0004	0.0075		
10	0	-0.0219	0.0000	0.0415	0.3716
10	1	0.0169	0.0054		
10	2	-0.0037	-0.0095		
10	3	-0.0048	-0.0161		
10	4	0.0083	0.0068		
10	5	0.0034	-0.0076		
10	6	0.0118	-0.0024		
10	7	0.0086	0.0035		
10	8	0.0027	0.0015		
10	9	-0.0020	0.0034		
10	10	-0.0097	-0.0022		

TABLE 3

OCEAN SOLUTION BASED ON LEVITUS SST (SET3)

MAXIMUM DEGREE OF EXPANSION = 36
NUMBER OF UNKNOWNS = 1369
RECIPROCAL CONDITION NUMBER = 0.134E-08
DETERMINANT = 4.054*10.0**4635

NUMBER OF SST VALUES USED = 301153
DEGREES OF FREEDOM = 28784
RMS FIT TO LEVITUS DATA = 0.01M

N	M	CNM	SNM	SN	SN(CUM)
0	0	0.4001	0.0000	0.4001	0.4001
1	0	0.1977	0.0000	0.5910	0.7137
1	1	0.5125	0.2180		
2	0	-0.8731	0.0000	1.1202	1.3283
2	1	0.2576	0.3219		
2	2	0.5136	0.2426		
3	0	-0.2745	0.0000	1.0768	1.7099
3	1	-0.6562	0.0622		
3	2	0.5007	0.2407		
3	3	0.1390	0.5673		
4	0	0.2345	0.0000	1.3236	2.1623
4	1	-0.7600	-0.2719		
4	2	-0.2409	-0.2406		
4	3	0.3914	0.4806		
4	4	-0.0454	0.7370		
5	0	0.5592	0.0000	1.4382	2.5969
5	1	-0.1638	-0.0862		
5	2	-0.8214	-0.6589		
5	3	0.0373	-0.1841		
5	4	-0.0193	0.5244		
5	5	0.0542	0.5468		
6	0	0.4162	0.0000	1.4254	2.9624
6	1	0.6958	0.1800		
6	2	-0.5612	-0.4501		
6	3	-0.4293	-0.6732		
6	4	-0.1661	-0.0907		
6	5	-0.0503	0.3137		
6	6	0.0519	0.2179		
7	0	-0.1648	0.0000	1.2118	3.2007
7	1	0.7265	0.2884		
7	2	0.2571	0.2226		
7	3	-0.2878	-0.4265		
7	4	-0.2723	-0.4710		
7	5	-0.1121	-0.1273		
7	6	-0.1660	0.2696		
7	7	-0.1561	-0.0255		
8	0	-0.2928	0.0000	1.1260	3.3930
8	1	0.2805	-0.0760		
8	2	0.5813	0.3121		

8	3	0.0837	0.0998		
8	4	-0.0954	-0.5187		
8	5	-0.0331	-0.4279		
8	6	-0.0247	0.0079		
8	7	-0.3193	0.1606		
8	8	-0.2158	-0.0902		
9	0	-0.2322	0.0000	1.0952	3.5653
9	1	-0.3324	-0.0399		
9	2	0.3492	0.0838		
9	3	0.1281	0.4949		
9	4	0.1157	-0.3580		
9	5	-0.0037	-0.5554		
9	6	0.2248	-0.2888		
9	7	-0.1672	0.0939		
9	8	-0.0331	0.0494		
9	9	-0.1213	0.0640		
10	0	0.1158	0.0000	1.0206	3.7085
10	1	-0.2882	-0.0943		
10	2	-0.1185	-0.3587		
10	3	-0.1696	0.5589		
10	4	0.1137	0.0414		
10	5	-0.0517	-0.4036		
10	6	0.2827	-0.3418		
10	7	-0.0207	-0.1467		
10	8	0.0746	-0.0531		
10	9	0.1409	0.0492		
10	10	-0.0488	0.1429		
11	0	0.1773	0.0000	0.7927	3.7923
11	1	-0.0220	0.0910		
11	2	-0.2388	-0.2856		
11	3	-0.2535	0.1834		
11	4	-0.0712	0.3351		
11	5	-0.0541	-0.0478		
11	6	0.2354	-0.1418		
11	7	0.1608	-0.1409		
11	8	-0.0430	-0.1739		
11	9	0.2526	-0.0254		
11	10	-0.0344	-0.0248		
11	11	0.0947	0.0255		
12	0	-0.2445	0.0000	0.8285	3.8818
12	1	0.3974	0.0672		
12	2	-0.0178	-0.1147		
12	3	-0.1595	-0.1500		
12	4	-0.0821	0.1711		
12	5	-0.0276	0.2169		
12	6	0.0632	0.0824		
12	7	0.4752	0.0615		
12	8	0.0126	-0.1290		
12	9	0.0874	-0.0288		
12	10	-0.1428	-0.0360		
12	11	-0.0543	-0.0702		
12	12	0.1495	-0.0374		
13	0	-0.5600	0.0000	1.0991	4.0344
13	1	-0.1256	-0.0220		
13	2	0.3153	0.4347		
13	3	0.2417	0.1112		
13	4	0.1825	-0.1929		
13	5	0.0708	0.0587		
13	6	-0.2434	0.1462		
13	7	0.3933	0.0389		

13	8	0.3083	0.0020		
13	9	0.0619	0.0068		
13	10	-0.1814	-0.0349		
13	11	-0.1608	-0.1641		
13	12	0.0799	0.0889		
13	13	-0.0483	-0.0261		
14	0	-0.1410	0.0000	1.4896	4.3006
14	1	-0.7008	-0.1921		
14	2	0.2002	0.1863		
14	3	0.2795	0.7613		
14	4	0.4308	0.0627		
14	5	0.3277	-0.2956		
14	6	-0.2647	-0.0425		
14	7	-0.1052	-0.2020		
14	8	0.2948	-0.0513		
14	9	0.2235	0.1067		
14	10	-0.1484	0.0308		
14	11	-0.0873	-0.3348		
14	12	0.2035	0.0788		
14	13	-0.0326	0.2180		
14	14	-0.2012	-0.0102		
15	0	0.7690	0.0000	1.8346	4.6755
15	1	-0.6433	-0.1638		
15	2	-0.2020	-0.2195		
15	3	0.2248	0.5732		
15	4	0.4069	0.6615		
15	5	0.3853	-0.2367		
15	6	0.0537	-0.3320		
15	7	-0.2872	-0.3256		
15	8	-0.0886	-0.3603		
15	9	0.1667	0.1460		
15	10	-0.0156	0.2967		
15	11	-0.1083	-0.2750		
15	12	0.4128	-0.2692		
15	13	0.0527	0.2832		
15	14	-0.2127	0.0514		
15	15	-0.1377	0.0382		
16	0	0.9291	0.0000	1.8639	5.0334
16	1	0.3775	0.0953		
16	2	-0.3550	-0.6428		
16	3	-0.1938	-0.3851		
16	4	0.2094	0.5918		
16	5	0.0596	0.3261		
16	6	0.1954	-0.3998		
16	7	0.0950	-0.2185		
16	8	-0.1285	-0.4780		
16	9	-0.1578	-0.0679		
16	10	0.1658	0.4254		
16	11	-0.2397	0.1705		
16	12	0.3792	-0.3115		
16	13	0.2557	0.0518		
16	14	-0.2179	0.0050		
16	15	-0.1642	-0.1460		
16	16	-0.0231	0.0295		
17	0	0.1095	0.0000	1.8368	5.3581
17	1	0.8762	0.1793		
17	2	0.0390	0.0258		
17	3	-0.2581	-1.0306		
17	4	0.1009	-0.4557		
17	5	-0.2689	0.4943		

17	6	-0.2565	-0.0833		
17	7	0.2932	-0.0427		
17	8	0.3655	-0.1353		
17	9	-0.2050	-0.1947		
17	10	0.1640	0.1408		
17	11	-0.0972	0.4322		
17	12	-0.0107	0.0441		
17	13	0.3555	-0.0294		
17	14	-0.1302	-0.0130		
17	15	-0.1085	-0.2368		
17	16	-0.1207	-0.1579		
17	17	0.0193	-0.0853		
18	0	-0.6225	0.0000	1.9150	5.6900
18	1	0.3064	-0.1169		
18	2	0.3324	0.4849		
18	3	-0.2019	-0.5177		
18	4	-0.0111	-1.0085		
18	5	-0.0045	-0.0263		
18	6	-0.7297	0.2449		
18	7	-0.1099	0.1858		
18	8	0.5869	0.2296		
18	9	0.1866	0.1515		
18	10	0.0412	-0.1700		
18	11	0.2244	0.1535		
18	12	-0.2985	0.2054		
18	13	0.0776	0.1253		
18	14	0.0177	0.0027		
18	15	0.0972	-0.0849		
18	16	-0.0402	-0.1960		
18	17	-0.1542	-0.1821		
18	18	0.0037	-0.1859		
19	0	-0.5070	0.0000	1.8826	5.9934
19	1	-0.4179	-0.5446		
19	2	-0.0305	0.4559		
19	3	0.1781	0.2104		
19	4	-0.1571	-0.5355		
19	5	0.4320	-0.4193		
19	6	-0.4774	0.2101		
19	7	-0.5405	0.4441		
19	8	0.1165	0.2386		
19	9	0.4289	0.5291		
19	10	0.0372	0.0261		
19	11	0.3434	-0.2393		
19	12	-0.1397	-0.0169		
19	13	-0.2519	0.1007		
19	14	-0.1024	0.0635		
19	15	0.1987	0.0111		
19	16	0.2763	0.1004		
19	17	-0.1296	-0.2036		
19	18	-0.0447	-0.2466		
19	19	-0.0282	-0.1690		
20	0	-0.0154	0.0000	1.5007	6.1784
20	1	-0.2340	-0.2969		
20	2	-0.4790	-0.1831		
20	3	0.0927	0.2876		
20	4	-0.2966	0.2657		
20	5	0.3525	-0.0315		
20	6	0.1592	0.0284		
20	7	-0.3299	0.5364		
20	8	-0.3663	0.1156		

20	9	0.1915	0.2489		
20	10	0.0745	0.4198		
20	11	0.2360	-0.1674		
20	12	0.0684	-0.1649		
20	13	-0.1757	-0.1657		
20	14	-0.2916	0.1075		
20	15	-0.0028	-0.0265		
20	16	0.3434	0.2648		
20	17	0.0573	0.0522		
20	18	-0.0173	-0.2859		
20	19	0.1269	-0.2238		
20	20	-0.0197	-0.0805		
21	0	-0.0506	0.0000	1.2860	6.3108
21	1	0.3113	0.1982		
21	2	-0.3128	-0.0442		
21	3	-0.0330	-0.1167		
21	4	-0.2236	0.1842		
21	5	-0.1674	0.4714		
21	6	0.2559	0.0690		
21	7	0.1617	0.4334		
21	8	-0.2455	0.1342		
21	9	-0.1385	-0.3629		
21	10	0.0369	0.2896		
21	11	0.1215	0.1661		
21	12	0.0023	0.0585		
21	13	0.0042	-0.2318		
21	14	-0.0649	0.0004		
21	15	-0.2765	0.0027		
21	16	0.0871	0.0736		
21	17	0.0531	0.2914		
21	18	0.0201	-0.1468		
21	19	0.2185	-0.2296		
21	20	0.1767	-0.1771		
21	21	0.0383	-0.0153		
22	0	-0.3011	0.0000	1.4688	6.4795
22	1	0.2021	0.4811		
22	2	0.3493	0.3100		
22	3	-0.1659	0.0245		
22	4	-0.0438	-0.3037		
22	5	-0.3548	0.3600		
22	6	-0.1164	0.1537		
22	7	0.1585	0.2818		
22	8	0.1650	0.2267		
22	9	-0.1728	-0.5124		
22	10	0.0350	-0.2220		
22	11	0.0187	0.1917		
22	12	-0.1310	0.3783		
22	13	-0.1998	-0.0244		
22	14	0.3242	-0.1162		
22	15	-0.1758	0.0940		
22	16	-0.1564	-0.1267		
22	17	-0.0982	0.1015		
22	18	-0.0320	0.1043		
22	19	0.1464	-0.0724		
22	20	0.2688	-0.1141		
22	21	0.1511	-0.1479		
22	22	0.0990	-0.0147		
23	0	-0.2809	0.0000	1.5811	6.6696
23	1	-0.3785	0.1171		
23	2	0.4915	0.3714		

23	3	0.1466	0.4822		
23	4	0.0901	-0.4813		
23	5	-0.1810	-0.2101		
23	6	-0.1341	-0.0585		
23	7	-0.2452	0.1594		
23	8	0.2589	0.2051		
23	9	-0.0559	-0.2085		
23	10	0.1196	-0.3113		
23	11	-0.0468	-0.0267		
23	12	-0.1149	0.3894		
23	13	-0.5706	0.1943		
23	14	0.2535	-0.1839		
23	15	0.2228	0.1156		
23	16	-0.1771	-0.0941		
23	17	-0.1327	-0.2160		
23	18	-0.0848	0.0515		
23	19	0.0776	0.1527		
23	20	0.0855	0.0779		
23	21	0.2091	-0.0850		
23	22	0.1382	-0.0827		
23	23	0.1117	-0.0383		
24	0	0.2486	0.0000	1.5724	6.8525
24	1	-0.6420	-0.0801		
24	2	0.1597	-0.2219		
24	3	0.1965	0.5273		
24	4	0.0956	-0.0175		
24	5	-0.0469	-0.4681		
24	6	0.2995	-0.4444		
24	7	-0.2988	0.0148		
24	8	0.0025	0.1241		
24	9	-0.0825	0.0055		
24	10	0.1499	0.0581		
24	11	-0.0727	0.0280		
24	12	-0.0736	0.1262		
24	13	-0.5923	0.2163		
24	14	-0.1742	-0.2858		
24	15	0.3892	0.0256		
24	16	-0.0263	0.0348		
24	17	-0.0424	-0.2438		
24	18	-0.0240	-0.2021		
24	19	0.0722	0.1649		
24	20	-0.0665	0.2193		
24	21	0.0156	0.0473		
24	22	0.1968	-0.0472		
24	23	0.1304	0.0048		
24	24	0.0680	-0.0381		
25	0	0.5157	0.0000	1.5372	7.0228
25	1	-0.2138	-0.0957		
25	2	-0.1969	-0.4966		
25	3	0.1790	-0.0886		
25	4	0.0627	0.2963		
25	5	-0.2004	-0.2646		
25	6	0.4424	-0.5225		
25	7	0.1613	-0.0811		
25	8	-0.1657	0.1256		
25	9	-0.2567	-0.0187		
25	10	0.0044	0.2284		
25	11	-0.1139	0.3629		
25	12	-0.1431	-0.0570		
25	13	-0.2887	0.0510		

25	14	-0.3337	-0.3871		
25	15	0.2658	-0.1470		
25	16	0.0894	0.1171		
25	17	0.0226	-0.0725		
25	18	0.0822	-0.2033		
25	19	0.1117	-0.0001		
25	20	-0.0602	0.2150		
25	21	-0.2069	0.0582		
25	22	0.0928	-0.0041		
25	23	0.1769	0.0499		
25	24	0.1164	0.0596		
25	25	0.0221	-0.0090		
26	0	0.4152	0.0000	1.4691	7.1748
26	1	0.2210	0.0444		
26	2	-0.1114	-0.2983		
26	3	0.0728	-0.5555		
26	4	0.0168	0.2596		
26	5	-0.2602	0.0637		
26	6	0.1186	-0.2616		
26	7	0.4793	-0.0431		
26	8	-0.0516	0.2451		
26	9	-0.3654	-0.0136		
26	10	-0.2097	-0.0096		
26	11	-0.1521	0.4593		
26	12	-0.2874	-0.0420		
26	13	-0.0436	-0.1694		
26	14	-0.1141	-0.3842		
26	15	0.2269	-0.2773		
26	16	0.0800	0.1306		
26	17	0.0130	0.0596		
26	18	0.0918	0.0089		
26	19	0.1329	-0.0409		
26	20	-0.0135	0.1115		
26	21	-0.2260	0.0160		
26	22	-0.0997	-0.0629		
26	23	0.1319	0.0290		
26	24	0.1075	0.1094		
26	25	0.0919	0.0622		
26	26	0.0082	0.0260		
27	0	0.0398	0.0000	1.3016	7.2919
27	1	0.2916	0.0135		
27	2	-0.0425	0.0866		
27	3	0.1574	-0.5089		
27	4	-0.0154	0.0440		
27	5	-0.1782	0.1167		
27	6	-0.2509	0.0329		
27	7	0.3366	0.0737		
27	8	0.0828	0.3573		
27	9	-0.2870	0.1286		
27	10	-0.3403	-0.2360		
27	11	-0.1134	0.1681		
27	12	-0.3522	-0.0206		
27	13	-0.0006	-0.2792		
27	14	0.1130	-0.3084		
27	15	0.3501	-0.2544		
27	16	0.0912	0.1454		
27	17	-0.0565	0.0919		
27	18	0.0488	0.0932		
27	19	0.0644	0.0799		
27	20	0.0161	0.0381		

27	21	-0.1341	-0.0209		
27	22	-0.1218	-0.1206		
27	23	0.0120	-0.0229		
27	24	0.0663	0.0569		
27	25	0.0504	0.1001		
27	26	0.0584	0.0413		
27	27	0.0177	0.0418		
28	0	-0.0362	0.0000	1.0513	7.3673
28	1	0.0770	0.0570		
28	2	-0.0901	0.1577		
28	3	0.1162	-0.2021		
28	4	-0.0861	0.1189		
28	5	-0.0637	0.0497		
28	6	-0.2935	0.1102		
28	7	0.0615	0.1103		
28	8	0.0161	0.3161		
28	9	-0.1522	0.2280		
28	10	-0.3155	-0.2205		
28	11	-0.0143	-0.1361		
28	12	-0.2344	-0.1160		
28	13	-0.0100	-0.2827		
28	14	0.1624	-0.2039		
28	15	0.3693	-0.1252		
28	16	0.1819	0.1998		
28	17	-0.1472	0.1124		
28	18	-0.0051	0.0191		
28	19	-0.0005	0.1089		
28	20	-0.0063	0.0525		
28	21	-0.0733	-0.0462		
28	22	-0.0205	-0.1019		
28	23	-0.0015	-0.0499		
28	24	0.0014	0.0176		
28	25	-0.0004	0.0502		
28	26	0.0188	0.0732		
28	27	0.0236	0.0304		
28	28	0.0217	0.0294		
29	0	-0.0284	0.0000	0.9017	7.4222
29	1	0.0628	0.1218		
29	2	-0.2552	0.1105		
29	3	0.0341	-0.1249		
29	4	-0.1302	0.2493		
29	5	-0.1254	0.0638		
29	6	-0.2003	0.0671		
29	7	-0.0165	0.0762		
29	8	-0.0991	0.1655		
29	9	-0.1163	0.1615		
29	10	-0.2020	-0.1457		
29	11	0.0564	-0.2045		
29	12	-0.0171	-0.1806		
29	13	0.0364	-0.2522		
29	14	0.1558	-0.0762		
29	15	0.2212	-0.0017		
29	16	0.2052	0.2360		
29	17	-0.1655	0.1651		
29	18	-0.0743	-0.0575		
29	19	-0.0064	0.0062		
29	20	-0.0396	0.0559		
29	21	-0.0520	-0.0381		
29	22	0.0255	-0.0824		
29	23	0.0707	-0.0249		

29	24	-0.0174	0.0122		
29	25	-0.0186	-0.0014		
29	26	-0.0151	0.0370		
29	27	-0.0036	0.0475		
29	28	-0.0032	0.0298		
29	29	0.0130	0.0098		
30	0	-0.0095	0.0000	0.8095	7.4663
30	1	0.1428	0.1920		
30	2	-0.2071	0.0990		
30	3	-0.1019	-0.2037		
30	4	-0.1168	0.2189		
30	5	-0.2072	0.1596		
30	6	-0.1509	0.0359		
30	7	0.0220	0.0233		
30	8	-0.0605	0.0509		
30	9	-0.1313	0.0288		
30	10	-0.1065	-0.1365		
30	11	0.0747	-0.1507		
30	12	0.1153	-0.1127		
30	13	0.0888	-0.1966		
30	14	0.1732	0.0094		
30	15	0.0697	0.0746		
30	16	0.1023	0.1920		
30	17	-0.1228	0.1780		
30	18	-0.1224	-0.0638		
30	19	0.0085	-0.0808		
30	20	-0.0153	0.0073		
30	21	-0.0454	-0.0057		
30	22	0.0091	-0.0729		
30	23	0.0978	0.0024		
30	24	0.0171	0.0316		
30	25	-0.0349	-0.0063		
30	26	-0.0053	-0.0200		
30	27	-0.0066	0.0295		
30	28	-0.0127	0.0260		
30	29	-0.0114	0.0272		
30	30	-0.0023	0.0008		
31	0	-0.1259	0.0000	0.6787	7.4970
31	1	0.1699	0.1561		
31	2	-0.0818	0.1606		
31	3	-0.1381	-0.2381		
31	4	-0.0461	0.0074		
31	5	-0.2242	0.1331		
31	6	-0.1562	0.0506		
31	7	0.0130	-0.0028		
31	8	0.0711	0.0189		
31	9	-0.0846	-0.0269		
31	10	-0.0519	-0.1370		
31	11	0.0638	-0.1228		
31	12	0.1207	0.0011		
31	13	0.0684	-0.0947		
31	14	0.1582	0.0297		
31	15	0.0145	0.1107		
31	16	-0.0093	0.1022		
31	17	-0.0815	0.1135		
31	18	-0.1024	-0.0443		
31	19	0.0121	-0.0888		
31	20	0.0250	-0.0279		
31	21	-0.0156	0.0129		
31	22	-0.0134	-0.0336		

31	23	0.0613	-0.0095		
31	24	0.0355	0.0514		
31	25	-0.0358	0.0107		
31	26	-0.0102	-0.0286		
31	27	0.0090	-0.0169		
31	28	-0.0017	0.0183		
31	29	-0.0139	0.0151		
31	30	-0.0074	0.0218		
31	31	-0.0097	0.0015		
32	0	-0.1394	0.0000	0.4894	7.5130
32	1	0.0499	0.0800		
32	2	0.0572	0.1696		
32	3	-0.1326	-0.1417		
32	4	0.0167	-0.1075		
32	5	-0.1442	0.0176		
32	6	-0.1175	0.0463		
32	7	-0.0458	-0.0149		
32	8	0.1150	0.0164		
32	9	0.0077	-0.0052		
32	10	-0.0118	-0.0883		
32	11	0.0471	-0.1125		
32	12	0.0903	0.0444		
32	13	0.0074	0.0028		
32	14	0.0837	0.0205		
32	15	0.0160	0.0986		
32	16	-0.0545	0.0436		
32	17	-0.0598	0.0233		
32	18	-0.0555	-0.0398		
32	19	0.0196	-0.0639		
32	20	0.0344	-0.0204		
32	21	0.0142	0.0152		
32	22	-0.0150	0.0021		
32	23	0.0228	-0.0134		
32	24	0.0253	0.0347		
32	25	-0.0203	0.0274		
32	26	-0.0253	-0.0150		
32	27	0.0106	-0.0255		
32	28	0.0110	-0.0036		
32	29	-0.0014	0.0071		
32	30	-0.0122	0.0059		
32	31	-0.0055	0.0107		
32	32	-0.0084	0.0044		
33	0	-0.1015	0.0000	0.3211	7.5199
33	1	-0.0346	-0.0010		
33	2	0.0717	0.1148		
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33	4	0.0438	-0.0979		
33	5	-0.0690	-0.0583		
33	6	-0.0556	0.0191		
33	7	-0.0699	-0.0077		
33	8	0.0626	0.0091		
33	9	0.0469	0.0132		
33	10	0.0176	-0.0191		
33	11	0.0271	-0.0817		
33	12	0.0667	0.0211		
33	13	-0.0188	0.0390		
33	14	0.0061	0.0175		
33	15	0.0151	0.0554		
33	16	-0.0448	0.0268		
33	17	-0.0397	-0.0164		

33	18	-0.0240	-0.0422		
33	19	0.0274	-0.0372		
33	20	0.0254	-0.0031		
33	21	0.0183	0.0122		
33	22	-0.0048	0.0118		
33	23	0.0020	-0.0033		
33	24	0.0162	0.0108		
33	25	-0.0091	0.0224		
33	26	-0.0214	0.0002		
33	27	-0.0024	-0.0152		
33	28	0.0132	-0.0093		
33	29	0.0054	0.0054		
33	30	-0.0051	0.0012		
33	31	-0.0075	-0.0002		
33	32	-0.0034	0.0008		
33	33	-0.0022	0.0026		
34	0	-0.0242	0.0000	0.1713	7.5218
34	1	-0.0613	-0.0257		
34	2	0.0530	0.0449		
34	3	-0.0365	-0.0041		
34	4	0.0373	-0.0367		
34	5	-0.0189	-0.0507		
34	6	-0.0006	-0.0076		
34	7	-0.0456	-0.0039		
34	8	0.0121	0.0075		
34	9	0.0256	0.0076		
34	10	0.0247	0.0199		
34	11	0.0098	-0.0350		
34	12	0.0377	-0.0013		
34	13	-0.0066	0.0232		
34	14	-0.0256	0.0111		
34	15	0.0057	0.0172		
34	16	-0.0193	0.0134		
34	17	-0.0155	-0.0120		
34	18	-0.0104	-0.0301		
34	19	0.0225	-0.0169		
34	20	0.0189	0.0079		
34	21	0.0027	0.0131		
34	22	-0.0007	0.0076		
34	23	-0.0051	0.0006		
34	24	0.0055	0.0030		
34	25	-0.0027	0.0083		
34	26	-0.0090	0.0049		
34	27	-0.0077	-0.0083		
34	28	0.0058	-0.0065		
34	29	0.0077	0.0014		
34	30	0.0015	0.0058		
34	31	-0.0048	-0.0021		
34	32	-0.0022	-0.0022		
34	33	-0.0024	-0.0016		
34	34	0.0006	0.0007		
35	0	-0.0072	0.0000	0.0737	7.5222
35	1	-0.0300	-0.0168		
35	2	0.0196	0.0089		
35	3	-0.0164	-0.0012		
35	4	0.0234	-0.0101		
35	5	-0.0092	-0.0231		
35	6	0.0065	-0.0068		
35	7	-0.0123	-0.0027		
35	8	-0.0012	0.0071		

35	9	0.0049	-0.0008		
35	10	0.0116	0.0184		
35	11	0.0037	-0.0044		
35	12	0.0106	-0.0030		
35	13	0.0045	0.0044		
35	14	-0.0194	0.0010		
35	15	-0.0013	0.0009		
35	16	-0.0033	0.0009		
35	17	-0.0014	-0.0023		
35	18	0.0000	-0.0102		
35	19	0.0103	-0.0050		
35	20	0.0109	0.0113		
35	21	-0.0031	0.0108		
35	22	-0.0036	0.0042		
35	23	-0.0053	-0.0024		
35	24	-0.0005	0.0003		
35	25	-0.0008	0.0020		
35	26	-0.0038	-0.0001		
35	27	-0.0021	-0.0036		
35	28	0.0001	-0.0055		
35	29	0.0053	-0.0006		
35	30	0.0029	0.0037		
35	31	0.0012	0.0031		
35	32	-0.0010	-0.0007		
35	33	-0.0001	-0.0008		
35	34	-0.0029	-0.0011		
35	35	0.0001	-0.0006		
36	0	0.0026	0.0000	0.0286	7.5222
36	1	-0.0138	-0.0056		
36	2	0.0081	-0.0015		
36	3	-0.0066	0.0015		
36	4	0.0093	-0.0019		
36	5	-0.0001	-0.0058		
36	6	0.0040	-0.0013		
36	7	-0.0016	-0.0043		
36	8	-0.0020	0.0046		
36	9	0.0004	-0.0020		
36	10	0.0016	0.0048		
36	11	0.0019	0.0025		
36	12	0.0018	0.0007		
36	13	0.0035	0.0015		
36	14	-0.0050	-0.0006		
36	15	-0.0021	-0.0012		
36	16	0.0005	-0.0023		
36	17	0.0013	-0.0013		
36	18	0.0025	-0.0014		
36	19	0.0018	-0.0010		
36	20	0.0055	0.0062		
36	21	-0.0026	0.0065		
36	22	-0.0031	0.0021		
36	23	-0.0029	-0.0013		
36	24	-0.0012	-0.0015		
36	25	-0.0001	0.0003		
36	26	-0.0002	-0.0003		
36	27	0.0002	-0.0014		
36	28	0.0008	-0.0030		
36	29	0.0010	-0.0010		
36	30	0.0022	0.0009		
36	31	0.0001	0.0019		
36	32	0.0006	0.0013		

36	33	-0.0002	0.0002
36	34	-0.0002	-0.0004
36	35	-0.0015	-0.0008
36	36	0.0000	-0.0014